Seminar 7: The Logic of *Principia Mathematica*

Volume 1 of *Principia Mathematica,* in which Russell and Whitehead set out their reduction of arithmetic to logic was published in 1910. Although the reduction adopts a conception of number very much like Frege's, it is developed in a system of logic designed to avoid the paradox that defeated Frege. Avoiding it required giving up the idea that for every formula in the logical language, there is a set (perhaps empty, perhaps not) of all and only the things that satisfy it. The problem was to block the generation of problematic sets while allowing those needed for the reduction.

Russell used a hierarchical theory of sets that was somewhat similar to Frege's hierarchy of concepts. His system can be formulated as one in which predicates designate, not Fregean concepts, but the extensions of those concepts (i.e. the sets of things of which the predicates are true)**.** On this picture, the members of 1st-level sets are individuals (i.e. non-sets), the members of 2nd-level sets 1st-level sets, and so on. Quantifiers predicate membership of an n-level set in an n+1-level set. First-order quantification (over individuals) predicates membership of a 1st-level set in a 2nd-level set (the extension of the quantifier). Second-order quantification (over 1st-level sets) predicates membership of a 2nd-level set in a 3rd-level set (the extension of the 2nd-order quantifier), and so on. Russell's paradox won't arise, because we can't express the idea of a set being a member of itself, or not being a member of itself.

Russell's conception of natural numbers – as sets of equinumerous sets – parallel Frege's conception of them as sets of concepts with equinumerous extensions. For Russell, zero is the 2nd-level set the only member of which is the first-level set that contains no members. The number 1 is the 2nd-level set that contains all and only 1st-level sets that contain an individual x and only x. The number 2 is the 2nd-level set that contains all and only 1st-level sets that contain distinct individuals x and y, and only them. In general, the successor of m is the set n of those 2nd-level sets z that contain a member which, when removed from z, leaves one with a member of m. The set of natural numbers is the smallest 3rd-level set that contains zero and is closed under successor (i.e. contains the successor of each of its members). If we don't run out of sets, this is sufficient to prove the axioms of arithmetic in a paradox-free system.

What is needed not to run out of 2nd-level sets to serve as natural numbers? We need infinitely many individuals (non sets). Suppose, for the sake of argument, there were only 10 individuals. Then the number 10 would be the 2nd-level set the only member of which is the 1st-level set containing all 10 individuals. The successor of 10 – i.e. 11 – would be the 2nd-level set of those 1st-level sets z that contain an individual x, which, when removed from z, leave one with the set of 10 individuals. But since there is no such 2nd-level set z, the successor of 10 – i.e. 11 – will turn out to be *the 2nd-level set with no members*. If we then compute the successor of 11, it too will be the 2nd-level set with no members. So in this scenario it turns out that two different numbers – 10 and 11 – have the same successor. Since this violates one of the axioms of arithmetic, Russell's reduction of arithmetic will fail unless there are infinitely many individuals (non-sets). Realizing this, he added it as an axiom of his system.

Doing this was technically, but not philosophically unproblematic. Since the existence of infinitely many individuals (non-sets) isn't a matter of pure logic, the need for the axiom defeats the classical epistemological motivation for logicism – namely to ground all mathematical certainty in the most fundamental and unchallengeable form of logical certainty. Realizing this, Russell found other justifications, including the unification of mathematics in a single comprehensive system and the ability to explain how mathematical knowledge could, in principle, be achieved from an underlying system of logic plus his hypothesis about the multiplicity of individuals. Later reductions were able to dispense with his need to assume infinitely many individuals by adopting different principles for generating sets – ones that prevented the generation of paradoxical sets by divorcing set theory from logic and setting it up as an axiomatic mathematical theory in its own right. What no one could do was reduce all mathematics to anything that could plausibly be regarded as pure logic.

Russell the Ontologist: The "No Class Theory"

This sketch fits the way Russell was understood by most logicians and scientifically minded philosophers in the decades that followed *Principia Mathematica*. Alfred Tarski, Rudolf Carnap, Carl Hempel, and W.V.O. Quine all saw his reduction as an important achievement, and took lessons from it about the importance of set theory for the development of logic and mathematics. But they didn't view his reduction in the way Russell did. By the time of *Principia Mathematica*, he had rejected sets, or "classes," which he now called "logical fictions." The higher-order variables in *Principia Mathematica* were said to range over "propositional functions" rather than sets. By 1910 he had rejected his earlier "realist" view of them as non-linguistic entities, and had come to think of them as simply formulas. In doing so, he adopted a de-ontologized interpretation of his technical reduction that was shared by virtually no one else. Most mathematicians, logicians, and philosophers regarded an ontology of sets as unproblematic. Over time, the preferred method of studying them often came not from a set theoretic version of Russell's theory of logical types, but from the axiomatic treatment of ZF set theory.

Although Russell allowed himself speak of "sets/classes" he explicitly disavowed commitment to them as entities. His position is sketched in section 2, Chapter 3 of the Introduction to *Principia Mathematica*.

> "The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their uses are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such symbols are so defined that when the *definiens* is substituted for the *definiendum*, there no longer remains any symbol supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals." (71-2)

According to Russell, a formula that seems to say that F is true of the (first-level) set of individuals satisfying G is really an abbreviation of a more complex formula that says that F is true of some *propositional function* that is true of all and only the individuals that satisfy G. Suppose, for the moment, that they are functions from objects to propositions. Then, to say that such a function is true of an object is to say that it assigns the object *a true proposition*, and to say that propositional functions are *extensionally equivalent* is to say they are true of the same things. (Similarly for two properties or for a property and a propositional function.)

Next consider a propositional function that takes a property or propositional function as argument. Call it *extensional* iff it whenever it is true of its argument A, it is true of all arguments extensionally equivalent to A. Not all propositional functions are extensional in this sense, but many are. Suppose that p_1 and p_2 are different but extensionally equivalent propositional functions, the former mapping an arbitrary individual a onto the proposition *that if a is a human, then a is a human* and the latter mapping a onto the proposition *that a if a is a featherless biped, then a is a human*. Now let 'Y' be a first-level predicate variable. Then the propositional function designated by ⌈I believe ∀xYx⌉ is one that maps propositional functions onto propositions expressed by the corresponding belief ascriptions. It may assign p_1 a true proposition about what I believe while assigning p_2 a false proposition. Thus, the propositional function designated by the belief ascription is *intensional*, rather than extensional.

Since, as Russell plausibly holds, *only* extensional propositional functions are relevant to mathematics, the system in *Principia Mathematica* can be restricted to them. When one does this, the only thing about the proposition assigned by a propositional function to a given argument that matters in Russell's system is its truth value. This being so, we can reinterpret the system to which he reduces arithmetic as one involving functions from arguments to truth values (rather than functions from argument to propositions) – *without losing anything essential to the reduction*.

In fact we can go further. A function from arguments to truth values is *the characteristic function of the set* of things to which it assigns truth. There is no mathematically significant difference between working with sets and working with their characteristic functions; anything done with one can be done with the other. Nor is there any important philosophical difference between the two. Why then did Russell insist on calling classes "logical fictions", thus denying that there really are such things? And how can one understand the system in *Principia Mathematica* if we take him seriously on this point?

Quantification in *Principia Mathematica*

The answer to the first question is that when classes are treated as real entities two basic problems connected with Russell's type-theoretic reduction of arithmetic to logic resist solution. The first is that Russellian natural numbers have to be located at some level of the hierarchy – most naturally the level at which we can predicate things of sets of sets of individuals. This makes it impossible for Russell to capture the insight that *any entities* -- including classes if there by such – in the way he wishes. He explains what this amounts to in a passage from *Introduction to Mathematical Philosophy* in 1919.

> "In seeking a definition of number, the first thing to be clear about is what we may call the grammar of our inquiry. Many philosophers, when attempting to define number, are really setting to work to define plurality, which is quite a different thing. *Number* is what is characteristic of numbers, as *man* is what is characteristic of men. A plurality is not an instance of number, but of some particular number. A trio of men, for example, is an instance of the number 3, and the number 3 is an instance of number; but the trio is not an instance of number. This point may seem elementary and scarcely worth mentioning; yet it has proved too subtle for the philosophers, with few exceptions.
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> A particular number is not identical with any collection of terms having that number: the number 3 is not identical with the trio consisting of Brown, Jones, and Robinson. *The number 3 is something which all trios have in common*, and which distinguishes them from other collections. A number is something that characterizes certain collections, namely, those that have that number." (11-12).

Russell's point is that just as the definition of *human* should provide us is not with an individual human but with the common characteristic of all humans, so the definition of the number three should provide us not with any set of three things, but with something that all sets of 3 things have in common – membership in the number three. Since all sets at all levels can be counted, Russell's goal of providing something all trios have in common can't be satisfied if 3 is located at one level of the hierarchy.

The second problem with taking classes to be entities was in giving non-arbitrary justifications of the type-theoretic restrictions of Russell's hierarchy. The hierarchy is designed to guarantee for each meaningful formula at every level of the hierarchy that there is a set of all and only those things that satisfy the formula. But in order to block paradox, certain formulas are ruled illegitimate. It is not enough for Russell that these restrictions be technical devices that block contradiction. *To be purely logical, they must be inherent in general principles for reasoning meaningfully about any subject.* One hint that his type restrictions don't meet that standard is that many statements that violate them appear not only to be meaningful but true. E.g., *For any two sets whatsoever, if their members can be put in 1 to 1 correspondence, they are equinumerous; No set is a member of itself; No set at one level of the hierarch is identical with any set at another level of the hierarchy; For each formula at every level of the hierarchy there is a set of all and only those things that satisfy it.* The apparent truth of all these claims, which violate the type restrictions, suggests that if sets are genuine entities we can talk about in the way we can talk about other things, then Russell's type theory *artificially restricts* what we can say about them. Worse, some things it doesn't allow us to say govern the construction of the hierarch itself.

For reasons like these, Russell wished to avoid commitment to classes altogether. By the time of *Principia Mathematica,* he thought he knew how to do so. The key was his understanding of quantification – which included elements of what we now call *ordinary objectual quantification,* elements of what we now call *metalinguistic quantification*, and elements of what we now call *substitutional quantification*. In Russell's time, these had not been clearly distinguished, so it isn't surprising that some of his comments in *Principia Mathematic* suit one of these, and some suite others, without any explicit recognition that the alternatives are quite different.

Consider the quantificational sentences: ∀x …x… and ∀Φ …Φ…

On an ordinary objectual interpretation these tell us that for every individual o and every class C …x… and …Φ… are true when the variables are assigned the object or class as value. It doesn't matter how large the domains of objects and classes are – finite, countably infinite (the size of the set of natural numbers), non-countably infinite (the size of the set of all subsets of natural numbers), and so on ascending to greater and greater infinities.

On a metalinguistic interpretation the consequences of these quantified sentences are metalinguistic claims that each substitution instance -- '…n…' or '….f…' *is a true sentence* – where a substitution instance is formed by erasing the quantifier and substituting either a name for the individual variable 'x' or a formula for the variable 'Φ'. Assuming that there are only countably many names and formulas in the language, the metalinguistic instances of these quantified claims are always countable.

The substitutional interpretation is like the metalinguistic interpretation except that the consequences of the quantified claims are all its substitution instances, rather than metalinguistic claims that those sentences are true. Russell's "no-class theory" requires thinking of quantification substitutionally.

In *Principia Mathematica* he speaks of propositions and propositional functions in a variety of different, and not always consistent, ways. But most of the time he seems to take propositions to be sentences, and propositional functions to be formulas one gets from them by replacing an occurrence of an expression with a free occurrence of a variable. Thus, in *An Inquiry into Truth and Meaning* (1940) he says "In the language of the second-order variables denote symbols, not what is symbolized," (192), while in *My Philosophical Development* (1959) he says "Whitehead and I thought of a propositional function as an expression." (92) If this really was what the authors meant, and if propositional functions were the values over which their higher-order variables ranged, it might seem that a sentence of the form '∀P …P…' must mean that *every value of the formula '…P…'* is true.

Language very like this is not hard to find in *Principia Mathematica*. For example, in section 3 of chapter 3 of the Introduction, Russell sketches the idea of a hierarchy of notions of truth that apply to the different levels of his type construction. Assuming that truth has already been defined for quantifier-free sentences at the lowest level, he explains first-order quantification as follows:

> "Consider now the proposition ⌈∀x Φx⌉. If this has truth of the sort appropriate to it, that will mean that every value of Φx has "first truth" [the lowest level of truth]. Thus if we call the sort of truth that is appropriate to ⌈∀x Φx⌉ "second truth," *we may define ⌈∀x Φx⌉ as meaning ⌈every value for 'Φx' has first truth⌉* … Similarly…*we may define ⌈∃x Φx⌉ as meaning ⌈some value for 'Φx' has first truth⌉.*

In addition to assuming that a similar explanation can be given for higher-order quantification, we assume that "first-truth" conditions and meanings have first been given for quantifier-free sentences.

For atomic sentences, the truth conditions are assumed to involve correspondence with atomic facts that consist of particular individuals standing in n-place relations. The truth conditions of sentences that are truth-functions of atomic sentences can be computed from the atomic facts. Supposing that propositional functions are formulas with free occurrences of variables, we can take the values of such formulas to be *sentences* that result from substituting an individual constant for all free occurrences of the variable in the propositional function. On this interpretation, his claim about what quantified sentences *mean* takes them to predicate truth of the (closed) sentences that are "values" of the corresponding open formulas that are identified with "propositional functions; universal generalizations say that each of these "values" is true, while existential generalizations say that some are.

However, this won't do. First, it would make all arithmetical statements arising from Russell's reduction to be metalinguistic statements about his logical language, with the result that all arithmetical knowledge would be characterized as knowledge of that language. That can't be correct. Second, it would drive an epistemological and metaphysical wedge between quantified statements and their instances. For when quantificational statements are understood in this way and ⌈Φn⌉ makes no claim whatever about language, it will be neither an a priori nor a necessary consequence of ⌈∀x Φx⌉ (if the semantic properties of Φx are neither essential to it nor knowable a priori), while ⌈∃x Φx⌉ will be neither an a priori nor a necessary consequence of ⌈Φn⌉ (on the same assumption). Third, it doesn't fit what Russell says four pages later in *Principia Mathematica* about the relationship between the facts in virtue of which universal generalizations are true and those in which their instances are.

> "We use the symbol ⌈∀x Φx⌉ to express the general judgment which asserts all judgments of the form Φx. … in any judgment ⌈∀x Φx⌉ the sense in which this judgment is or may be true is not the same as that in which Φx is or may be true. If Φx is an elementary judgment, it is true when it *points to* a corresponding complex [i.e. to a fact that makes it true]. But ⌈∀x Φx⌉ does not point to a single corresponding complex [i.e. there is no single fact that makes it true]: the corresponding complexes [facts] are as numerous as the possible values of x." (46)

If ⌈∀x Φx⌉ *meant* that all substitution instances of the formula Φx were true (in the appropriate sense of 'truth'), then the condition necessary and sufficient for its truth *would be* the existence of a general fact about language, and not the existence of the multiplicity of nonlinguistic facts corresponding to all its instances (which seems to be what Russell has in mind). Moreover, the metalinguistic judgment expressed by ⌈∀x Φx⌉ would not, as Russell indicates that it does, straightforwardly "assert" all the nonmetalinguistic judgments expressed by its instances. For these reasons, the metalinguistic interpretation of quantification suggested by some of Russell's remarks wasn't his consistent and considered view (if he had one).

There is a better interpretation. On this interpretation, the quantifiers in his reduction are what are now called "substitutional." They don't range over objects of any kind – linguistic or nonlinguistic. Instead they are associated with substitution classes of expressions. Although their *truth conditions* are stated metalinguistically, *their content* is supposed to be nonlinguistic. Thus, they are not subject to the objections just raised against the metalinguistic interpretation of the quantifiers. Using objectual quantifiers over expressions, we can give substitutional *truth conditions* of quantified sentences in the normal way – as Russell does. ⌈∀x Φx⌉ and ⌈∃x Φx⌉ are true, respectively, iff all, or some, of their substitution instances are true, where the latter are gotten from replacing free occurrences of 'x' in Φx by an expression in the relevant substitution class. *This explanation will work, provided that the truth values of the sentences on which the quantified sentences depend are already determined before reaching the quantified sentences, and so do not themselves depend on the truth or falsity of any higher-level substitutionally quantified sentences.*

There are three important points to note. First, if one combines the hierarchical restriction inherent in substitutional quantification with Russell's system of higher levels of quantification, strong versions of the type restrictions he needs will fall out from the restrictions on substitutional quantification, without requiring further justification. Second, on the substitutional interpretation, there is no need for what look like "existential" generalizations – i.e. ⌈∃x Φx⌉, ⌈∃P Φ(P)⌉, ⌈$\exists P_2$ $\Phi(P_2)$⌉, etc. – to carry any ontological commitment. They won't – as long as the relevant substitution instances can be true even when the constant replacing the bound variable doesn't designate anything. *Third, for this reason, it is tempting to think that no quantificational statements in the hierarchy carry any ontological commitments not already carried by quantifier-free sentences at the lowest level. Since Russell took accepting their truth to commit one only to individuals and properties, it would be natural for him to take himself to be free to characterize classes, numbers, and nonlinguistic propositions and propositional functions as merely "logical fictions," while nevertheless appealing to them when "speaking with the vulgar."*

Let's look at this a little more closely. Suppose that Russell's first-order quantification is substitutional, and so depends on quantifier-free sentences at the first level of his hierarchy. These sentences will include all truth-functional compounds of the atomic sentences (which consist of n-place predicates combined with n occurrences of names). Predicates stand for n-place properties, while names designate individuals. An atomic sentence is true iff its names designate individuals that have the property designated by its predicate. Since Russell's axiom of infinity requires infinitely many individuals (non-sets), a substitutional interpretation of first-order quantification will require *infinitely many names* that can be substituted for individual variables in order to secure all the instances needed to evaluate the quantified sentence. On this interpretation, $\ulcorner\forall x\ \Phi x\urcorner$ will be true iff every sentence is true that results from substituting an occurrence of a name associated with 'x' for each occurrence of 'x' in Φx; $\ulcorner\exists x\ \Phi x\urcorner$ is true iff at least one such substitution instance is true.

Second-order quantification occurs at the next level of the hierarchy. Here predicate are variables associated with predicates of the first level. The associated predicates include all *simple predicates* used to construct atomic sentences, plus *complex predicates*. *For any first-level sentence in which simple predicates occur, we need a complex predicate for each of the ways of abstracting one or more of the predicates via lambda abstraction – as illustrated by expressions like $\lambda F\lambda G[\Phi\ (\ldots F\ldots G)]$.* All these predicates, simple and compound, are associated with the predicate variables. So, on the substitutional interpretation, $\ulcorner\forall X_1\ \Phi(X_1)\urcorner$ is true iff every sentence is true that results from substituting an occurrence of a predicate, simple or complex, associated with 'X_1' for each occurrence of 'X_1' in $\Phi(X_1)$; similarly for second-order existential quantification.

Looking at this from the outside (where we allow ourselves to speak of sets), this means that our substitutional construal of second-order quantification parallels ordinary objectual second-order quantification over *those sets that are extensions of first-level predicates of individuals* (including complex predicates). This process is repeated for third-order quantification, except that here complex predicates are the only ones in the substitution class. This level mimics objectual quantification over *those sets that are extensions of second-level predicates, members of which are sets of individuals that are extensions of first-level predicates*. The hierarchy continues uniformly from there on.

We now know that this results in a huge diminishment of expressive power of higher-order quantification. Whereas objectual quantifiers range over *all sets at a given level* – both those that are extensions of predicates at that level (of Russell's language) and those that aren't -- the substitutional quantifier mimics only quantification over the former. If, as is standardly assumed, every sentence and every predicate is a finite sequence of the logical and nonlogical vocabulary, the domain of *all sets at a given level* will far outstrip the domain of all sets that are the extensions of predicates at that level. As a result, the expressive power of the underlying "logical" theory to which arithmetic is to be reduced is drastically diminished by treating its quantifiers substitutionally, to the detriment of Russell's reduction.

Because these results hadn't been established in 1910, there was no way for Russell to know them. To him, the substitutional conception – to the limited extent that he could distinguish it from the normal objectual conception – seemed to solve his problems. Recall the problem of locating the reduction at one particular level of the hierarchy. On the ontological interpretation of higher-order quantification, locating the numbers at a single level makes it impossible for Russell's definition of the number 3 to capture what all trios have in common – namely membership in the number 3 -- because lots of trios of classes at higher levels are excluded from membership. *But if one thinks there are no classes, and indeed no entities of any sort beyond the individuals and properties designated by quantifier-free sentences at the first level, the problem evaporates; all trios of entities will be members of the number 3.* Second, if quantifiers never range over totalities of entities at all – because they are substitutional – there is no need to restrict the legitimate totalities over which they are allowed to range. On the contrary, the restrictions inherent in quantification fall out of the very meaning of the substitutional quantifiers. Thus, one who takes all quantification to be substitutional will view the type restrictions imposed by Russell's hierarchical system to be inherent in all intelligible thought.

Seeing this removes what would otherwise be a puzzle in the interpretation of Russell. On the one hand his original theory of types in *Principia Mathematica* – called the "ramified theory of types" -- was highly complicated, very restrictive, and required an axiom -- called the "axiom of reducibility" – that sparked distrust and disbelief from the moment it was formulated. The ramified theory and its associated axiom was given up by virtually all mathematicians and logicians, including Russell, after 1926 when Frank Ramsey replaced the ramified theory with the *simple theory of types*. For these metamathematicians, who thought of quantification as objectual, the new theory was a vast improvement over the original. But the *philosophical idea* that Russell's logicist reduction showed how the ontological commitments associated with a given area of discourse could be drastically reduced by logical means retained a very strong following in philosophy for decades, especially during the heyday of logical positivism, between 1925 and 1945. What was not then recognized is that the ontological idea – and in particular his "no class theory" -- was linked in Russell to his technically impoverished, and I would say inadequate substitutional interpretation of quantification. Thus, if we want to preserve the mathematical legacy of *Principia Mathematica* we need to clearly separate it from some of the philosophically suspect lessons Russell and other philosophers drew from it.

In section 5 of chapter 10 of my book, between pp. 520 and 531, I catalogue some of the main problems to which Russell's flirtation with substitutional quantification led. *For example, between 520 and 525, I show how it blocked the most natural Fregean proof of mathematical induction while making it impossible to imagine reductions of mathematical theories that allow uncountably many objects*. Between pp. 525 and 531 I show that it was also inconsistent with various parts of his general philosophical logic, including his account of quantification in "On Denoting." According to "On Denoting" the proposition *that everything is F* predicates *always assigning a truth when given an object as argument* of the function that maps an argument o onto the proposition that predicates F-hood of o. This quantified proposition isn't equivalent to any collection of propositions, finite or infinite: *that* o_1 *is F, that* o_2 *is F, ….* For any such collection C, it is possible for the general proposition to be false even if all the particular propositions in C are true. (Just imagine possible scenarios in which there are more individuals, of the relevant sort, than are covered by the individuals in C.) Also, knowing that everything is F doesn't guarantee knowing any instance of that claim; though one who knows the general proposition, while also knowing of o, has enough to conclude that o is F.

Russell didn't repudiate these thoughts, so congenial to the usual objectual understanding of first-order Frege-Russell quantification, when writing Volume 1 of *Principia Mathematica*. Granted, some of the passages in that work do suggest a substitutional interpretation. However other passages, like the following, don't.

> "Our judgment that all men are mortal collects together a number of elementary judgments. It is not, however, composed of these, since (e.g.) *the fact that Socrates is mortal is no part of what we assert, as may be seen by considering the fact that our assertion can be understood by a person who has never heard of Socrates. In order to understand the judgment "all men are mortal," it is not necessary to know what men there are*. We must admit, therefore, as a radically new kind of judgment, such general assertions as "all men are mortal." (45)

On a substitutional interpretation, the truth of 'All men are mortal' should, for Russell, consist in the truth of 'If Socrates is a man, then Socrates is mortal,' 'If Plato is a man, then Plato is mortal,' etc. Putting this together with the passage from *Principia* leads to an obvious question. How can one who has never encountered the name 'Socrates' or heard of the man it names (and similarly for other men and names) make the judgment, or even understand the sentence 'All men are mortal?"

This question is easily answered, if the quantification is objectual in the usual Frege-Russell sense. On this interpretation, the proposition expressed by the sentence says of a certain (real, non-linguistic) function f_{mortal} -- from arguments to truth (in Frege's case) or from arguments to truths (in Russell's case) -- that it *assigns truth (or a truth) to an argument iff that argument is a man*. Although entertaining this proposition, and knowing it to be true, requires acquaintance with f_{mortal} and the property predicated of the function, it does not require acquaintance with any particular individual, much less knowledge of which individuals are men, or even what individuals there are. Hence, the objectual understanding of the quantifier fits Russell's observation in the passage.

The same can't be said of the substitutional understanding of the quantifier. On that account "All men are mortal" must be seen as meaning something like *If Socrates is a man, then Socrates is mortal, If Plato is a man, then Plato is mortal, and so on*. This suggests that understanding the substitutional interpretation of the quantified sentence requires understanding the names 'Socrates', 'Plato', and so on – which for Russell requires *being acquainted with the man Socrates, the man Plato, and so on*. The same holds for all names in the substitution class. Far from vindicating the observation in the last quoted above passage, this result contradicts it. Thus, if first-order quantification in *Principia* is substitutional, then the account of quantification in *Principia* is inconsistent with some of Russell's metatheoretical statements about it there, as well as with the account of quantification in "On Denoting," which underwrites those statements.

Although it hardly seems possible for things to get worse for the substitutional interpretation of Russell, they do. To avoid all commitment to classes, Fregean concepts, and non-linguistic propositional functions, he would have to treat even first-order quantification as substitutional. But then, since his axiom of infinity will require infinitely many individuals, he will need *infinitely* many *logically proper names*. For Russell, these are simple terms the meanings of which are their referents – with which we must be acquainted in order to understand the names. It is highly doubtful that anyone's language could contain infinitely many such terms. Surely, they couldn't all be learned as separate lessons, which is how one imagines they would have to be acquired. It is similarly doubtful that anyone could be acquainted, in Russell's highly restrictive sense, with infinitely many individuals. If, as presumably we must, assume (i) that no one is capable of such understanding and acquaintance, while recognizing (ii) that understanding the substitutional quantification employed requires it, we arrive at an interpretation of Russell's system that renders it unintelligible, *by his own lights*.

In response, one might say that a reasonable interpreter should revise some of Russell's restrictive views about names and acquaintance in the service of arriving at a more adequate version of his overall position. Perhaps, but if one takes this route, there are compelling reasons to include his skepticism about classes and his flirtation with substitutional quantification – which he was not then in any position to fully understand – as prime candidates for revision. The point is reinforced when we bring identity into the mix. The analysis of singular definite descriptions in "On Denoting" tells us that a formula, $\ulcorner\Psi$ (the x: Φx)$\urcorner$, containing a description is an abbreviation of the formula $\ulcorner\exists x\ \forall w\ [(\Phi(w) \leftrightarrow w = x)\ \&\ \Psi(x)]\urcorner$ containing the identity predicate. In *Principia Mathematica,* Russell's definition of identity (between individuals) tells us that the formula '$w = x$' is an abbreviation of the higher-order formula $\ulcorner\forall\Theta\ (\Theta w \leftrightarrow \Theta x)\urcorner$. Putting the two together we have the result that $\ulcorner\Psi$ (the x: Φx)$\urcorner$ is an abbreviation of $\ulcorner\exists x\ \forall w\ [(\Phi(w) \leftrightarrow \forall\Theta\ (\Theta w \leftrightarrow \Theta x))\ \&\ \Psi(x)]\urcorner$.

This is no problem if the predicate variables range over all subsets of the domain of individuals (or over their characteristic functions). Since these include all sets that contain only a single individual, x and w will be members of the same sets, and so satisfy the same predicates iff they are identical. But if quantifiers are interpreted substitutionally, then (since '=' isn't primitive) *there will be no guarantee that for every individual in the domain, there will be a first-order formula that is true of it, and nothing else*. There will be no guarantee that two or more different objects won't satisfy precisely the same first-order formulas, and so be indistinguishable in the system. Suppose there are such individuals. Then, what Russell's axiom of infinity (needed for the proof of arithmetical axiom 4) will require is not just infinitely many individuals, but *infinitely many individuals distinguished from one another by quantifier-free first-order formulas*. Worse, if distinct individuals x and y are *not* distinguished by the formal system, then any formula true of x will be true of y -- even one which (on the ordinary objectual reading of quantification) says *that x is the only member of a certain set*, or one that says of x *that, along with a distinct z*, are all and only the members of a member of the number two (the set of pairs of individuals).

How might these problems be avoided? *One could add infinitely many primitive predicates of individuals, each applying to a single individual and no two such predicates applying to the same individual*. But Russell didn't do that. Nor would he have done so, since it would, in effect, make knowledge of the arithmetical system *derived* from what was supposed to be *logic* dependent on understanding infinitely many primitive predicates of individuals -- in violation of his doctrine that one can understand propositions of logic without knowing any non-logical vocabulary. Thus, in addition to weakening his logicist program, the uniform substitutional interpretation of the hierarchy creates a problem for his definition of identity and conflicts with the standard formulation of his theory of descriptions, which he continued to employ in *Principia*.

In light of all these problems, it is, I think, a mistake to read a uniformly substitutional account of quantification into Russell's logicist program. It is true that some more or less inchoate thoughts of a substitutional sort played a role in his views about how paradoxes are to be avoided, how his type restrictions might be justified, and how classes, non-linguistic propositions, and non-linguistic propositional functions might be eliminated. The powerful attraction of eliminating what he saw as problematic entities and his need to see type-restrictions as conditions on the very intelligibility of seeming quantification over classes were powerful motivators pushing him toward a substitutional view of the quantifiers and a ramified theory of types. Nevertheless a reconstruction of his position that systematically treats quantification as substitutional rather than objectual creates worse problems than it solves – for both his logicist program and his broader philosophy.

That Russell himself didn't see these problems is due, in part, to the fact that he was not in a position to understand substitutional quantification as fully as we do today. Prior to the 1960s and 1970s few, if any, philosophers clearly recognized and distinguished substitutional from objectual quantification. What's more, fundamental metalogical and metamathematical results distinguishing 1st and 2nd-order arithmetic in terms of the power of 2nd-order quantification over sets -- were still many years in the future when *Principia Mathematica* was written. It's no shame that Russell wasn't aware of these things. What would be a shame is to saddle *Principia Mathematica* with an interpretation which, if consistently carried through, would obliterate or obscure much of the progress he made there.

The best interpretation is, I think, the one that best coheres with Russell's most important philosophical views, best advances his understanding of the relationship between logic and mathematics, and best explains the impact of his work on those who followed. Such an interpretation should, I believe, dismiss his radical eliminativism about classes and his flirtation with substitutional quantification as regrettable but understandable errors, while treating the quantification in his hierarchy as objectual, ranging over individuals and classes (or non-linguistic propositional functions). The complex ramified theory of types and the Axiom of Reducibility should be dropped in favor of the simple theory of types, through which most of the historical influence of Russell's reduction has flowed. I believe that it is this (relatively standard) interpretation that has the best chance of illuminating the strengths and weaknesses of his logicist program, while making intelligible its impact on later philosophers and logicians.