Overview of *The Rise of Analytic Philosophy*

Analytic philosophy didn't begin as a self-conscious revolt against earlier western philosophy. It began with interest in new topics – logic, language and mathematics – that hadn't been rigorously pursued before. The tradition started in 1879 when Frege invented modern logic with the aim of explaining how we are able to achieve certainty in mathematics. His strategy was to reduce higher mathematics to arithmetic, which was a process that was already underway, and then to reduce arithmetic to logic. To do this he had to develop a logic more powerful than any logical systems deriving from antiquity, which were still in use. *The fact that his logical framework can be applied to spoken human languages, doubled the achievement*. For Frege, the function of language is to represent the world. For a sentence S to be meaningful is for S to represent the world as being a certain way – which is to impose conditions it must satisfy if S is to be true. That idea provided the basis for a general theory of linguistic meaning.

Frege's philosophy of mathematics tried to answer two questions: *What is the source of mathematical knowledge?* and *What are numbers?* He answered that *logic* is the source of mathematical knowledge, 0 is the set of concepts true of nothing, 1 is the set of concepts that are true something, and only that thing, and so on. Since the concept *being my daughter* is true of nothing, it is an element of 0; since the concept *being chairman of USC philosophy* is true of me, and only me, it is an element of 1; since the concept *being my son* is true of Greg and Brian Soames and only them, it is an element of the number 2. Multiplication is defined as repeated addition, addition is defined as repeated counting, and counting is defined as an operation on sets. Thus, arithmetic was derived from what Frege took to be logic.

Frege's system contained a contradiction found by Bertrand Russell in 1903, after which he inherited the task of reducing arithmetic to logic. He completed that task in *Principia Mathematica*, using a more complicated version of Frege's ideas. Although he was mathematically successful, the complications he had to introduce were philosophically costly. *Frege dreamed of deriving mathematics from self-evidently obvious logical truths, but some of Russell's complications were neither obvious nor truths of logic*. Later reductions reduced the complications, but the systems to which they reduced mathematics were not logical systems that govern reasoning about all subjects. They were versions of an elementary mathematical theory now called set theory.

Despite this, *Principia Mathematica* illustrated the power of logical analysis to address philosophical problems. *In "On Denoting," Russell achieved success by arguing that the logical forms of our thoughts are often disguised by the grammatical forms of sentences we use to express them*. Following *Principia Mathematica* he applied this idea in *Our Knowledge of the External World,* 1914, and *The Philosophy of Logical Atomism,* 1918. There, he used basic concepts to state axioms and definitions from which central parts of our everyday and scientific knowledge could be derived. *The trick was to assign ordinary and scientific claims truth conditions that don't require anything we can't know*. In this way, he hoped, he could refute philosophical skepticism by showing how we can have knowledge of the world.

G.E. Moore, was Russell's undergraduate friend at Cambridge in the 1890s and his professional colleague in the early 20ᵗʰ century. Like Russell, he was concerned with skepticism and knowledge. For him, our knowledge of commonsense certainties is the starting point in philosophy – *e.g. the certainty that we are conscious beings with physical bodies inhabiting, with similar beings, a universe larger and older than we are*. Since any theory of knowledge must build on this starting point, no skeptical theory that denies it can be correct. *Unlike Russell, who revised the contents of commonsense convictions to make them consistent with a skeptical theory of knowledge, Moore retained the ordinary contents of our convictions and required philosophical analyses of knowledge to be consistent with them*.

Ethics was his other main interest. *He argued that although goodness is the central concept of ethics, it is undefinable, and, because of this, no ethical theory can be proven, or even supported by evidence*. These claims dominated ethics for decades, while his 3ʳᵈ claim, *that some important ethical truths are nevertheless knowable*, was widely rejected, leading to the rise of ethical non-cognitivism.

Russell and Moore turned British philosophy away from *Absolute Idealism*--a doctrine that held that Reality is spiritual and that every part of Reality is essential to every other part. Moore argued that Idealists confused objects of perception with our awareness of them; *e.g. they confused the blue we see with our consciousness of blue*. Using one word 'sensation' for both, they concluded that the world we perceive can't exist without being perceived. Moore corrected this error and offered a theory of mind in which all cognitions relate us to things outside us. He used Russell's theory of logical form to show the logical flaw in the Idealists' argument that every part of reality is essential to every other part.

With this an earlier school of speculative philosophy was replaced by the new *analytic* approach. *The main contributions of Frege, Moore, Russell were, the invention of modern logic, its use in the philosophy of mathematics, a theory of meaning that can be applied to all language, the use of logical analysis as a philosophical tool, the conception of science and common sense as philosophical starting points, and the detachment of ethics from other areas of philosophy*. Despite these innovations, the traditional aim of philosophy--*to achieve knowledge of the most important truths about the world*--remained intact.

That changed in 1922, when *Wittgenstein's Tractatus* ushered in an era in which philosophical problems were seen as linguistic problems to be solved by analyzing language. Whereas Russell was driven by a vision of what *reality* must be like if it is to be *knowable*; Wittgenstein was driven by a vision of what *thought* and *language* must be like if they are to *intelligibly represent* reality. Russell analyzed *conceptual* connections as *logical* connections; Wittgenstein reduced metaphysical and epistemic possibility to logical possibility. *Russell believed the aim of philosophy was to discover logical truths and definitions which, when applied to statements of mathematics, science, and everyday life, would reveal their true contents; Wittgenstein believed there are no philosophical truths, and also no truths of ethics, aesthetics, or religion*. For Wittgenstein, a sentence that is neither a tautology nor a contradiction has meaning only if its truth, or its falsity, is guaranteed by elementary facts. *Thus, he thought, there are no unanswerable questions and no inherently mysterious propositions*. Anything about which we can speculate is a topic of scientific inquiry. Since philosophy isn't a science, its job was restricted to clarifying thought and language.

Frege's Logical Language

Names stand for objects. Predicates stand for *concepts*, which are functions that assign truth or falsity to objects depending on whether they satisfy certain conditions. A simple sentence is true iff the *concept* designated by the predicate assigns truth to the *object* designated by the name. The language has ways of negating, conjoining, and disjoining sentences, and of forming conditional and biconditional sentences. Each way of forming a complex sentence involves a truth functional particle – 'not', 'and', 'or', 'if, then' and 'iff'. These designate functions that map truth values onto truth values. E.g., 'It is not the case that Lima is a city' is false because the function designated by negation maps the value *truth* designated by 'Lima is a city' onto the value *falsity*. The situation is similar with 'and', 'or', 'if, then' and 'iff', which designate functions that map pairs of truth values onto single truth values.

A complex predicate is formed by replacing occurrences of a name in a sentence S (whether S is simple or complex). Let S_n be a sentence '…n…n…' containing two occurrences of 'n' and let S_x be result from replacing them with the variable 'x'. S_x is a *formula* with a single free variable. It counts as a *predicate* that takes a single argument. S_x designates a *concept* C_{Sx} from objects to truth values. C_{Sx} assigns *truth* to objects o that make S_x true *when x is treated as a temporary name for o*. When o satisfies this condition, we say that C_{Sx} *is true of o*.

Quantifiers over individuals are higher-order predicates of 1st-order concepts (designated by 1st-order predicates). '∃x' designates the *2nd-order concept* that assigns *truth* to all 1st-order concepts that are *true of at least one object*; it *assigns* falsity to all other 1st-order concepts. '∀x' designates the 2nd-order concept that assigns *truth* to every 1st-order concept that is *true of all objects;* it *assigns* falsity to all other first-order concepts. Thus, ⌈∃x S_x⌉ is true iff C_{Sx} is true of some object and ⌈∀x S_x⌉ is true iff the complex predicate C_{Sx} is true of every object.

Like quantifiers, the definite description operator 'the x' designates a function, f_{the}, that assigns a value to 1st-order concepts. Unlike quantifiers, the values of f_{the} are objects rather than truth values. 'The x' attaches to S_x to form a *complex singular term 'the x: S_x'* that designates the one and only object of which C_{Sx} is true; if there isn't exactly one object of which C_{Sx} is true, f_{the} is undefined for C_{Sx} and 'the x: S_x' doesn't refer to anything.

The quantification we have talked about so far is 1st-order, which means it is quantification involving variables ranging over objects. Frege's system also allows quantifiers to combine with predicate variables that range over concepts.

Frege's Higher-Order System of Quantification

<u>1st-Order Quantification</u>
'∃x' designates the 2nd-order concept that takes a 1st-order concept C as argument and assigns it the value *truth* iff C assigns *truth* to at least one *object*. So ⌈∃x Φx⌉ is true iff the concept designated by Φx assigns *truth* to at least one object iff there is at least one object o such that using 'x' in Φx as a temporary *name* for o would make Φx true. E.g., '∃x (Number x & Even x & Prime x)' is true iff the concept *being a number which is both even and prime* is true of at least one object.

<u>2nd-Order Quantification</u>
'∃P' designates the 3rd-order concept C_3 that takes a 2nd-order concept C_2 as argument and assigns it the value *truth* iff C_2 assigns *truth* to at least one 1st-order *concept*. So ⌈∃P Φ(P)⌉ is true iff the 2nd-order concept designated by Φ(P) assigns *truth* to at least one 1st-order concept iff there is at least one 1st-order concept C such that using 'P' in Φ(P) as a *predicate* designating C would make Φ(P) true. E.g., '∃P (P Aristotle & P Plato & ~P Pericles)' is true iff the 2nd-level

concept *being a concept that is true of Plato and Aristotle but not Pericles* is true of at least one 1st-level concept iff there is some concept that is true of both Plato and Aristotle but not Pericles.

Similar rules give us higher-order universal quantification. But Frege's system doesn't stop with 2nd-order quantification. For all n, his system allows nth-order quantification over n-1 order concepts. E.g., call any 1st-order concepts A and B *equal* iff the objects of which A is true can be paired off 1-1 with the objects of which B is true. This 2-place predicate, '( ) is equal to ( )', combines with a pair of *1st-order predicates* to form an atomic formula, e.g. 'A is equal to B'. The new predicate designates a *2nd-order concept* that assigns *truth* to a pair of 1st-order concepts iff the objects of which one member of the pair is true can be paired 1-1 with the objects of which the other is true. Next we replace 'is equal to' with a 2-place, *2nd-order predicate variable* R^2 that ranges over *2nd-order concepts*. This gives us the formula 'A R^2 B' that designates the 3rd-order concept C_3 that assigns *truth* to any 2nd-order concept C_2 that assigns *truth* to the pair of 1st-order concepts A and B. The existential quantifier '$\exists R^2$' designates a 4th-order concept C_4 that assigns a 3rd-order concept *truth* iff that concept assigns *truth* to *at least one 2-place, 2nd-order concept*. Thus, the quantified sentence '$\exists R^2$ [A R^2 B]' is true iff the 4th-order concept designated by the quantifier assigns *truth* to the 3rd-order concept C_3 designated by the formula. So, the quantified sentence will be true iff *there is at least one 2nd-level concept that is true of the 1st-level concepts A and B*. This is *third-order quantification over second-order concepts*.

The Relationship between Frege's Conception of Logic and his Conception of Mathematics

Frege's interest in mathematics focused on two questions: *"What are numbers?"* and *"What is the basis of mathematical knowledge?"* He was convinced that the highest certainty belongs to elementary, self-evident principles of logic – *without which thought itself might prove impossible*. Thus, he believed that the certainty of arithmetic and higher mathematics (save geometry), must be deductively based on logic itself. It was to demonstrate this that he developed modern logic. The key step after that was to derive arithmetic from logic by (i) specifying a small set of logical truths of the highest certainty to serve as axioms, (ii) defining all arithmetical concepts in terms of purely logical ones, and (iii) producing formal proofs of all arithmetical axioms from these definitions plus the axioms of logic.

This program was due, in part, to his muscular conception of logic. His logic carried its own ontology. An infinitely ascending hierarchy of predicates was matched by an infinitely ascending hierarchy of concepts they denoted. First-level concepts were functions from objects to truth values, second-level concepts were functions from first-level concepts to truth values, and so on. *In addition, every concept had an extension (itself taken to be a kind of object), which we may regard as the class of entities (possibly empty) to which the concept assigned the value truth.* Frege's "logical axioms" guaranteed the existence of infinitely many entities of this sort. Today many would say that his logic looks like set theory, which is widely regarded not as logic *per se*, but as a fundamental mathematical theory in its own right. But that is hindsight.

Frege's Approach to the Philosophy of Mathematics

The key to Frege's philosophy of mathematics was his methodology. Although prior to philosophical analysis we all know many arithmetical truths, we have no idea what numbers are and little understanding of how it is possible for us to achieve certain knowledge of them. *His idea is that natural numbers are whatever they have to be in order to explain our knowledge of them*. So the way to discover what they are and how we know statements about them is to frame definitions of each number, and of *natural number*, that allow us to deduce what we pretheoretically know. How, for example, should 2, 3, 5, and addition be defined so that facts like those in (2) can be deduced from the definitions, plus our knowledge of logic plus empirical facts like (1)?

1. ∃x ∃y (x is a black book on my desk & y is a black book on my desk & x ≠ y & ∀z (z is a black book on my desk → z = x or z = y)) & ∃u ∃v ∃w (u is a blue book on my desk & v is a blue book on my desk & w is a blue book on my desk & u ≠ v & u ≠ w & v ≠ w & ∀z (z is a blue book on my desk → z = u or z = v or z = w)) & ∀x∀y ((x is a black book & y is a blue book) → x ≠ y)

2a. The number of black books on my desk = 2 and the number of blue books on my desk = 3. (There are exactly 2 black books on my desk and exactly 3 blue books on my desk.)

  b. The number of books on my desk = 5. (There are exactly 5 books on my desk.)

More generally, how might a proper understanding of what natural numbers and arithmetical operations are be used first to derive our purely arithmetical knowledge from the laws of logic, and then to derive empirical applications of that knowledge by appealing to relevant empirical facts? This is the most important question a philosophical theory of number must answer.

Frege's chief objection to earlier philosophers of mathematics was that they didn't answer this question and what they did say got in the way of a proper answer. His chief target was John Stuart Mill, who held that our knowledge of the arithmetical truth *that 5+2 = 7* is derived from many empirical truths of the sort illustrated by the proposition *that the seven coins on the table are five coins belonging to Mary and two belonging to John*. Frege objected that this can't be so because the proposition *that 5+2 = 7* is necessary and known *a priori,* while the propositions from which Mill hoped to derive them are contingent and knowable only *a posteriori*. Contrary to Mill, our knowledge of such empirical truths is derived from our a priori knowledge of the truth *that 5+2 = 7* plus our empirical knowledge *that the number of coins on the table belonging to Mary = 5 and the number of coins on the table belonging to John = 2*.

More generally, Frege argued that Mill's view -- that our arithmetical knowledge is *justified* by the inductive support it receives from our sense experience -- reversed the proper order of explanation. Such experience supports a proposition p only if it raises the probability that p is true. But in order for the experience to support p, the revision of p's probability must obey the axioms of probability theory – e.g. when the probabilities of p and q are independent, the probability of their disjunction is the sum of their separate probabilities. More generally, assignments of probabilities based on evidence requires numerical calculations – *which means that the claim that certain propositions are supported (rendered probable) by evidence presupposes arithmetic*, which must already be justified. Thus, Mill's claim that arithmetic is inductively justified is self-defeating.

Frege makes a closely related point in the following passage,

> "It is in their nature to be arranged in a fixed, definite order of precedence; and each one is formed in its own special way and has its own unique peculiarities, which are specially prominent in the cases of 0, 1, and 2. Elsewhere when we establish by induction a proposition about a species, we are ordinarily in possession already, merely from the definition of the concept of the species, of a whole series of its common properties. *But with the numbers we have difficulty in finding even a single common property which has not actually to be first proved common.* … The numbers are literally created, and determined *in their whole natures*, by the process of continually increasing by one. Now, this can only mean that from the way in which a number, say 8, is generated through increasing by one *all its properties can be deduced. But this is in principle to grant that the properties of numbers follow from their definitions, and to open up the possibility that we might prove the general laws of numbers from the method of generation which is common to them all.*" (pp. 15-16, *Foundations of Arithmetic*)

Here, Frege argues that arithmetic doesn't inductively depend on experience because, when we understand what the natural numbers are and discover their proper definitions, we will see that the purely arithmetical propositions about them are logically derivable from those definitions alone, leaving nothing for experience to inductively justify.

Frege's Reduction of Arithmetic to Logic

<u>Concepts, Objects, and Numbers</u>

For Frege, *a statement of number is a statement about a concept* – which he illustrates by observing that the same totality may truly be described as consisting of one forest or 100 trees. So, for him, the statement that *there are 4 moons of Jupiter* is the statement that *the number of things falling under the concept **moon of Jupiter** = 4*. But since the identity predicate is flanked by singular terms, the truth of the statement requires those terms to designate the same *object*. So, although statements of number are about concepts, numbers themselves must be objects. Which objects?

Frege imposes a criterion of correctness on any proper answer. The criterion, known as *Hume's Principle,* states: *for all concepts F and G, the number belonging to F = the number belonging to G iff the extension of F (the set of things falling under F) can be put in 1-1 correspondence with the extension of G* – iff one can exhaust the set of things falling under F and the set of things falling under G by forming pairs the first falls under F and the second of which falls under G, where no member of either set occurs (in the same position) in more than one pair. This correspondence can be defined in non-numerical terms as follows.

> For all concepts F and G, the extension of F (the set of things falling under F) corresponds 1-1 with the extension of G (the set of things falling under G) iff for some relation R, (i) for every object x such that Fx, there is an object y such that Gy & Rxy, and for every object z if Gz & Rxz, then z=y, and (ii) for every object y such that Gy, there is an object x such that Fx & Rxy, and for every object z if Fz & Rzy, then z=x.

When two extensions (sets) correspond 1-1, they are called *equinumerous*, when the extensions of two concepts are equinumerous the concepts are called *equal*. So, for all concepts F and G, the number belonging to F = the number belonging to G iff F *equals* G iff the extension of F is *equinumerous* with the extension of G. When $\mathcal{F}$ and $\mathcal{G}$ are predicates designating concepts F and G, ⌈the number of $\mathcal{F}$s = the number of $\mathcal{G}$s⌉ is true iff the extension of F is equinumerous with the extension of G.

<u>Definitions</u>

Frege defines *the number belonging to the concept F* (informally, *the number of F's*).

*Def. For any concept F, the number of F's is the extension of the concept **equal to F**.*

The extension of a concept is the set of things that fall under it. For any concept F, the things falling under the concept *equal to F* are those the extensions of which are equinumerous with the extension of F. So, the number of F's is the set of all and only those concepts the extensions of which are equinumerous with the set of things of which F is true. E.g., the number of fingers on my right hand is the set of concepts the extensions of which are equinumerous with the set of fingers on my right hand – it is the set of concepts true of 5 things. We can see this is so, but we are not yet allowed to use a numeral like '5', since no definitions have yet been given for them.

*Def: Zero is the number that belongs to the concept **not identical with itself**.*

Zero is the set of concepts the extensions of which can be put into 1-1 correspondence with the set of things that aren't identical with themselves. Since there are no objects that are not identical with themselves, zero is the set of concepts the extensions of which correspond 1-1 with the empty set – the set with no members. So, zero is the class of concepts that don't apply to anything. (Since functions are identical iff they map the same arguments onto the same values, the only member of zero is the function that assigns *falsity* to all objects.)

Next we define the notion of n *directly following* (i.e., succeeding) m.

*Def: n directly follows (succeeds) m iff for some concept F, and object x falling under F, n is the number belonging to F, and m is the number belonging to the concept **falling under F but not identical with x**.*

So, n directly follows (succeeds) zero iff for some concept F, and object x falling under F, n is the number belonging to F, and zero (the set of concepts that aren't true of anything) is the number of the concept *falling under F but not identical with x*. Since this concept isn't true of anything, n is a set of concepts each of which is true of some object x, and only x. Assuming that there is just one such set, we have identified *the unique object* that directly follows (succeeds) zero.

Def: The successor of m is the unique object that directly follows (succeeds) m.

Numerals: '1' designates the successor of 0, '2' designates the successor of 1, etc. *So, 0 is the set of concepts under which nothing falls, 1 is the set of concepts under which, some x, and only x, falls, 2 is the set of concepts under which some non-identical objects x and y, and only they, fall, etc.* It is transparent that 0 is the set of concepts true of nothing, 1 is the set true of just one thing, 2 is the set true of just two things, and n is the set of concepts true of exactly n things. We don't presuppose a prior understanding of numerals, the result is simply a consequence Frege's definitions.

　　　Two features of the definitions:

 (i) Just as redness isn't identical with any red thing, but is something all red things have in common, so the number n is not identical with any set of n things, or with any concept that is true of exactly n things; it is something to which all those concepts bear the same relationship – membership.

(ii) Just as counting a group of things consists in putting them in 1-1 correspondence with the numerals used in the count, so 1-1 correspondence is the key to defining each number.

Finally, we define *natural number* in a way that allows us to prove the axiom *mathematical induction*.

Mathematical Induction
If zero falls under a concept, and a successor of something that falls under a concept always falls under the concept, then every natural number falls under the concept.
$\forall P \, [(P0 \ \& \ \forall x \forall y \, ((Px \ \& \ Sxy) \rightarrow Py)) \rightarrow \forall x \, (NNx \rightarrow Px)]$

Def. of Natural Number: Let an inductive concept be one that is true of zero and closed under successor (i.e., is true of the successor of x whenever it is true of x). For all x, (x is natural number iff all inductive concepts are true of x).

Proof of Mathematical Induction using this definition: Given our definitions, Mathematical Induction says that every inductive concept is true of all natural numbers. Since an inductive concept is one that satisfies the antecedent of Mathematical Induction, and since the definition of natural number guarantees that any such concept is true of all natural numbers, *it follows that Mathematical Induction is true*.

Frege's definition of Natural Number is different from, but equivalent to, the one just given. To understand it, it is useful to begin with the relations *parent* and *ancestor*: Since any parent of x is an ancestor of x, as is any parent of any ancestor of x, the *ancestor* relation is called the

transitive closure of the *parent* relation. We can generalize, using T to define what it is for R to be transitive and TC to define its transitive closure R_{TC}.

> T. $\forall x \forall y \forall z \ ((Rxy \ \& \ Ryz) \rightarrow Rxz)$
> TC. $\forall x \forall y \ (R_{TC}xy \ \text{iff} \ \forall P \ [(\forall z \ (Rxz \rightarrow Pz) \ \& \ \forall u \forall v \ ((Pu \ \& \ Ruv) \rightarrow Pv)) \rightarrow Py])$

Parent isn't transitive, but *ancestor* is, When *Rxy* is the relation *y is a parent of x*, $R_{TC}xy$ is the relation *y is an ancestor of x*. Going left to right in TC: if y is an ancestor of x, then, for any concept P, P will be true of y, if (i) P is true of every parent of x, and (ii) P is true of any parent of someone of whom P is true; and (right to left in TC) if for any concept P whatsoever, P is true of an individual y provided that, (i) P is true of every parent of x, and (ii) P is true of any parent of someone of whom P is true, then – given all this – y must be an ancestor of x. Thus, the transitive closure of an intransitive relation R is often called *the ancestral of R*. Frege used the ancestral of a relation to establish mathematical induction.

Frege's Proof of Mathematical Induction: Let *Rxy* be the relation that holds between any extension of a concept and its successor – using Frege's definition of successor (i.e. *directly follows*). The ancestral of this relation is the relation *y follows x (in a series closed under successor)*; in other words, *y is greater than x*. We then define the natural numbers as those sets (extensions of concepts) that are greater than or equal to zero. To prove Mathematical Induction, we assume that zero falls under P, and that P is closed under successor, and show that P must be true of every natural number. The antecedent of the axiom *P(0) & $\forall u \forall v((Pu \ \& \ Ruv) \rightarrow Pv)$* (in which *Rxy* is the successor relation, *y directly follows x)*. Since *greater than* is the ancestral, R_{TC}, of successor, R, TC tells us that P is true of everything greater than zero. Since, by definition, the natural numbers are zero plus everything greater than zero, it follows that every natural number falls under P.

Logical and Arithmetical Axioms

The first part of Frege's system of logical proof consists of axioms and inference rules for proving standard logical truths in the sense, recognized today, of formulas that are true on all interpretations of their non-logical symbols, and all choices of domains of quantification. Frege's system for proving such truths was as effective as any we have today. His system also included other "logical principles." Since meaningful predicates/formulas denote concepts that determine the objects they are true of, the comprehension and extensionality principles for concepts are taken for granted.

  Concept Comprehension: For every stateable condition φ on things, there exists a concept C that is true of all and only those things that satisfy the condition: $\exists C \ \forall y \ (Cy \leftrightarrow \phi y)$

  Extensionality: Concepts P and Q are identical iff everything that falls under one falls under the other. $\forall P \forall Q \ (P = Q \ \text{iff} \ \forall x \ (Px \leftrightarrow Qx))$

These principles plus special "logical" axiom V guarantee extensionality and comprehension for sets.
  Axiom V: For all (first-level) concepts P and Q the extension of P (the set of things falling under P) = the extension of Q (the set of things falling under Q) iff $\forall x \ (Px \leftrightarrow Qx)$.

This gives us *comprehension for sets* since, '$\forall P \forall x \ (Px \leftrightarrow Px)$' will always be true, which, by V, means that the set of things falling under P = the set of things falling under P. This doesn't require the set to be nonempty, but it does require there to be a set of all and only those things of which P is true. We also have *extensionality for sets* – identifying sets with the same members.

The axioms of *Peano Arithmetic* that Frege derived from his logic are as follows.

Peano Arithmetic:

A1 Zero isn't a successor of anything. $\sim \exists x\ Sx0$

A2 Nothing has more than one successor. $\forall x \forall y \forall z\ ((Sxy\ \&\ Sxz) \rightarrow y = z)$

A3 No two things have the same successor: $\forall x \forall y \forall z\ ((Sxy\ \&\ Szy) \rightarrow x = z)$

A4 Zero is a natural number: NN0

A5 Every natural number has a successor: $\forall x\ (NNx \rightarrow \exists y\ Sxy)$

A6 A successor of a natural number is a natural number: $\forall x \forall y\ ((NNx\ \&\ Sxy) \rightarrow NNy)$

A7 Mathematical Induction: If zero falls under a concept, and a successor of something that falls under a concept always falls under the concept, then every natural number falls under the concept. $\forall P\ [(P0\ \&\ \forall x \forall y\ ((Px\ \&\ Sxy) \rightarrow Py)) \rightarrow \forall x\ (NNx \rightarrow Px)]$

Informal Proofs

A1 states that there is no concept F, set of concepts x, and object y falling under F such that zero is the number belonging to F and x is the number belonging to the concept *falling under F but not identical with x*. Proof: Since zero is the set the only member of which is the concept under which nothing falls, the extension of that concept can't be put into 1-1 correspondence with the extension of any concept F under which something falls. So, zero can't be the number belonging to F.

A2: In order for x to have non-identical successors y and z, there would have to be concepts F and G such that (i) y = the number belonging to F, z = the number belonging to G, and y ≠ z, and (ii) F is true of some object o_F, G is true of some object o_G, x = the number belonging to the concept *falling under F but not identical with o_F* and x = the number belonging to the concept *falling under G but not identical with o_G*. This could be true only if the extensions of the concepts F and G could *not* be put into 1-1 correspondence, but the results of removing a single item from each could be put into such correspondence. Since this is impossible, A2 is true; the successor relation is a function.

A3: For two different things x and y to have the same successor z there must be concepts F and G and objects o_F and o_G such that (i) z = the number belonging to F = the number belonging to G, (ii) x = the number belonging to the concept *falling under F but not identical with o_F* and y = the number belonging to the concept *falling under G but not identical with o_G*, and (iii) x ≠ y. (i) tells us that the extensions of F and G can be put in 1-1 correspondence, while (ii) and (iii) tell us the extensions that result from removing a single item from each can't be put into such correspondence. Since this is impossible, A3 is true. Given A2, A3 tells us that successor is 1-1.

By the definition of *natural number*, A4 says that zero falls under the concept *equals zero or follows zero in a series under successor*. Hence A4 is true by definition.

A5: We start with zero, which is the number belonging to the concept *not identical with itself*. This is the set of concepts under which nothing falls, i.e. the set the only member of which is the concept that assigns falsity to every argument. The successor of zero is the number belonging to a concept F under which an object x falls, such that the concept *falling under F but not identical with x* is a member of zero. Can we be sure, on the basis of Frege's logic, that there is such a concept F and object x? Yes. We already have zero, and we know there is a concept *being identical with zero*. This plus Axiom V guarantees that there is a set which is the number belonging to this concept, and hence that there is a set of concepts the extensions of which are equinumerous with the set the only member of which is zero. This set of concepts under which exactly one thing falls is the successor of zero – i.e., the number 1. Since 1 *follows zero in the series under successor*, it is a natural number. Similar reasoning – this time using the concept

being identical with either with zero or 1 – establishes that 1 has a successor – the number 2 – which is the set of concepts under which a pair of non-identical things and nothing else, fall. Since 2 also follows zero in the series under successor, it is also a natural number. So every instance of A5 is derivable using Frege's logic plus definitions. This isn't itself a proof of the universal generalization A5, but Frege found a way of turning it into one. With this, we are assured that the successor function is totally defined on the natural numbers.

A6: If x is a natural number, x is zero or x follows zero in the series under successor. A5 ensures x has a successor – which must follow zero in the series under successor and so be a natural number.

A7, mathematical induction, follows trivially from the definition of *natural number*.

Arithmetical Operations

Peano arithmetic defines the arithmetical operations *addition* and *multiplication*.

Definition of Addition: For any natural numbers x and y, the result of adding zero to x is x; the result of adding the successor of y to x is the successor of the result of adding y to x.
$\forall x \, \forall y \, [(NNx \ \& \ NNy) \rightarrow (x + 0) = x \ \& \ (x + \$(y)) = \$(x + y)]$

This *recursive* definition first specifies what it is to add zero to an arbitrary number x, and then specifies what it is to add the successor of a number y to a number x. We first determine the sum of zero and x to be x. We then determine the sum of x and the successor of zero (namely 1) to be the successor of x. Applying the definition again, we determine the sum of x and the successor of 1 (namely 2) to be the successor of the successor of x. The process determines x+y, for each number y. Since x can be any number, the definition determines the sum of every pair of numbers.

Illustrative derivation: 3 + 2 = 5.

(i) $\$(\$(\$(0))) + \$(\$(0)) = \$[\$(\$(\$(0))) + \$(0)]$
         From Def. of '+' and A4,A6, which guarantee that $\$(\$(\$(0)))$ and $\$(\$(0))$ are natural numbers
(ii) $\$(\$(\$(0))) + \$(0) = \$[\$(\$(\$(0))) + 0]$ From Definition of '+', A4, and A6
(iii) $\$(\$(\$(0))) + \$(\$(0)) = \$(\$[\$(\$(\$(0))) + 0])$ From (i) and (ii) by substitution of equals for equals
(iv) $\$(\$(\$(0))) + 0 = \$(\$(\$(0)))$ From the Definition of '+'
(v) $\$(\$(\$(0))) + \$(\$(0)) = \$(\$(\$(\$(\$(0)))))$ From substitution in (iii) on the basis of (iv)

To show that Frege's logical system allows the derivation of results such as these, one must show his logical axioms guarantee there is a (unique) function f that satisfies the pair of equations defining addition: *f(x,0) = x* and *f(x,$(y)) = $(f(x,y))* Given the strength of Frege's comprehension and extensionality principles plus Axiom V, this is not problematic.

Analogous results hold for multiplication, which is defined in a similar fashion.

Definition of Multiplication: For any natural numbers x and y, the result of multiplying x times zero is zero, and the result of multiplying x times the successor of y is the sum of x and the result of multiplying x times y. $\forall x \, \forall y \, [(NNx \ \& \ NNy) \rightarrow ((x * 0) = 0 \ \& \ x * \$(y) = (x * y) + x)]$

Frege on Sense, Reference, Compositionality, Hierarchy

Frege's argument that meaning (sense) is distinct from reference is based on sentences that differ only by substituting one coreferential term for another, as in (1-3).

1a.   The brightest heavenly body visible in the early evening sky (at certain times and places) is the same size as the brightest heavenly body visible in the morning sky just before dawn (at certain times and places).

  b.   The brightest heavenly body visible in the early evening sky (at certain times and places) is the same size as the brightest heavenly body visible in the early evening sky (at certain times and places).

2a.   Hesperus is the same size as the brightest heavenly body visible in the morning sky just before dawn (at certain times and places).

  b.   Hesperus is the same size as Hesperus.

3a.   Hesperus is the same size as Phosphorus.
  b.   Phosphorus is the same size as Phosphorus.

Frege's contention that the (a)/(b) sentences differ in meaning is intended to explain three facts. (i) one can understand both sentences without taking them to mean the same thing, or to agree in truth value. (ii) one who assertively utters (a) would typically be taken to say something different from what one would say by assertively uttering (b). (iii) one would typically use the (a) and (b) sentences in ascriptions ⌜A believes that S⌝, in which (a)/(b) take the place of S, to report different beliefs. *If* these three points are sufficient for the (a) and (b) sentences to differ in meaning, then principles T1 and T2 can't be jointly maintained.

T1. The meaning of a name or a definite description is the object to which it refers.

T2. The meaning of a sentence S (or other compound expression E) is a function of its grammatical structure plus the meanings of its parts; hence, substituting an expression β for an expression α in S (or E) will result in a new sentence (or compound expression) the meaning of which does not differ from that of S (or E), provided that α and β do not differ in meaning.

Frege rejects T1. For him, their meanings aren't their referents. Instead, meaning, or in his terminology *sense*, is the mode by which the referent of a term is presented to one who understands it. This sense, or *mode of presentation,* is a condition satisfaction of which by an object o is necessary and sufficient for o to be the referent of the term. Although different terms with the same sense have the same referent, terms designating the same referent may differ in sense. Frege takes this to explain the difference in meaning between the (a)/(b) not only in (1) and (2), but also in (3).

The case of proper names is complicated by the fact that speakers commonly use the same name to refer to the same thing, even though they think of it differently. For Frege, this suggests that the sense of a name n, as used by a speaker s at a time t, to be a reference-determining condition that could, in principle, be expressed by a description. On this view, n as used by s at t, refers to o iff o is the unique object that satisfies the descriptive condition associated with n by s at t. Thus, for Frege, (3a) and (3b) differ in meaning for any speaker who associates the two names with different descriptive modes of presentation.

For Frege, all terms have senses distinct from their referents. This is true, *even though (4a), (4b), and the identity statement '$6^4 > 2375$' are a priori truths of arithmetic that qualify as "analytic" for Frege*.

4a. $6^4 > 1296$
  b. $1296 > 1295$

This means that two expressions '6^4' and '1296' can have different Fregean senses despite the fact that it is possible for one who understands both to reason a priori from knowledge *that for all x '6^4' refers to x iff x = 6^4 and for all y '1296' refers to y iff y = 1296* to the conclusion *that '6^4' and '1296' refer to the same thing*.

Although Frege rejects T1, he accepts principles of compositional sense, T2, and compositional reference, T3, for terms, as well as thesis T4 about sentences.

T3. The referent of a compound term E is a function of its grammatical structure, plus the referents of its parts. Substitution of one coreferential term for another in E (e.g. substitution of '5^3' for '125' in 'the successor of 125') results in a new compound term ('the successor of 5^3') the referent of which is the same as that of E. If one term in E fails to refer, then E does too (e.g., 'the successor of the largest prime').

T4. The truth or falsity of a sentence is a function of its structure, plus the referents of its parts. Substitution of one coreferential term for another in a sentence S results in a new sentence with the same truth value as S. So the following pairs are either both true, or both false.

> The author of the *Begriffsschrift* was widely acclaimed during his time.
> The author of "On Sense and Reference" was widely acclaimed during his time.

> The probe penetrated the atmosphere of Hesperus.
> The probe penetrated the atmosphere of Phosphorus.

> $2^{10} > 6^4$
> $1024 > 1296$

Noticing that the truth values the sentences we have looked at have all depended on the referents of their parts, Frege subsumed T4 and T5 under T3 by holding that sentences refer to truth values – the True and the False – which he took to be objects of a certain kind. Thus, he took T4 and T5 to be corollaries of T3.

T5. If one term in a sentence S fails to refer, then S is neither true nor false.

> The present king of France is (isn't) wise.
> The largest prime number is (isn't) odd.

On Frege's account of negation, ⌈~S⌉ lacks a truth value when S does because there is no argument on which the truth function designated by the negation operator can operate. The analysis generalizes to many-place predicates and truth functional connectives. Reference failure anywhere in a sentence results in its truth valuelessness. Such sentences aren't epistemically neutral, however, because the norms governing belief/assertion require truth.

For Frege, the referent (truth value) of a sentence is *compositionally determined* by the referents of its parts, while its meaning (the thought it expresses) is *composed* of the meanings of its parts. Just as (5) consists of a subject phrase and a predicate, the *thought* it expresses, consists of the sense of the subject and the sense of the predicate

5.   The author of the *Begriffsschrift* was German.

He generalizes:

> "If, then, we look upon thoughts as composed of simple parts, and take these, in turn, to correspond to the simple parts of sentences, we can understand how a few parts of sentences can go to make up a great multitude of sentences, to which, in turn, there correspond a great multitude of Thoughts."

Being a Platonist about senses, Frege believed that there is such a thing as *the* meaning of 'is German', and that different speakers who understand the predicate know that it has that meaning. For him, senses are public objects available to different thinkers. There is one thought – *that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the remaining sides* – that is believed by all who believe the Pythagorean theorem. It is this that is preserved in translation, and this that is believed or asserted by agents who sincerely accept, or assertively utter, a sentence synonymous with the one used to state the theorem. For Frege, thoughts and their constituents are abstract objects, imperceptible to the senses, that are grasped by the intellect.

Just as the sense of a definite description may be taken to be a condition the unique satisfaction of which by an object is sufficient for that object to be the referent of the term, so a thought expressed by a sentence may be taken to be a condition the satisfaction of which by the world as a whole is sufficient for the sentence refer to the True. *The strain in the analogy comes when one considers what happens when no object uniquely satisfies a description, as opposed to what happens when the world as a whole doesn't "satisfy" the thought expressed by a sentence*. In the former case, the description lacks a referent, while in the latter case Frege takes the referent of the sentence to be a different object, the False

This brings us to an important complication. Frege recognized that, given the compositionality of reference principle, he had to qualify his view that sentences refer to truth values. While taking the principle to unproblematically apply to many sentences, he recognized that it doesn't apply to *occurrences of sentences* as content clauses in attitude ascriptions ⌈A asserted/ believed that S⌉. Suppose, for example, that (6a) is true, and so refers to the True.

6a Jones believes that 2+3 = 5.

Since '2+3 = 5' is true, substituting any other true sentence – e.g., 'Frege was German' – for it ought, by T3, to give us another true statement, (6b), of what John believes.

6b. Jones believes that Frege was German.

But this is absurd. An agent can believe one truth (or falsehood) without believing every truth (or falsehood). Thus, if the truth values of attitude ascriptions are functions of their grammatical structure, plus the referents of their parts, then the complement clauses of such ascriptions must, if they refer at all, refer to something other than the truth values of the sentences occurring there.

Frege's solution to this problem is illustrated by (7), in which the putative object of belief is indicated by the italicized noun phrase.

7.   Jones believes *the thought expressed at the top of page 91*.

Since the phrase is not a sentence, its sense is not a thought. Thus, what is said to be believed – which is itself a thought – must be the referent of the noun phrase that provides the argument of 'believe', rather than its sense. This result is generalized in T6.

T6 The thing said to be believed in an attitude ascription ⌈A believes E⌉ (or similar indirect discourse report) is what the occurrence of E in the ascription (or report) refers to.

Possible values of 'E' include ⌜the thought/ proposition/claim that S⌝, ⌜that S⌝, and S. In these cases what is said to be believed is the thought that S expresses. If T6 is correct, this thought is the *referent* of *occurrences* of S, ⌜that S⌝, and ⌜the thought/ proposition/ claim that S⌝ in attitude ascriptions (or other indirect discourse reports). So, in an effort to preserve his basic tenets – that meaning is *always* distinct from reference, and that the referent of a compound is *always* compositionally determined from the referents of its parts – Frege was led to T7.

T7. An occurrence of a sentence S embedded in an attitude ascription (indirect discourse report), refers not to its truth value, but to the thought S expresses when it isn't embedded. In these cases, an occurrence of S refers to S's ordinary sense. Unembedded occurrences of S refer to the ordinary referent of S – i.e. its truth value.

Here, Frege takes not expressions but their *occurrences* to be semantically fundamental. Unembedded occurrences express "ordinary senses," which determine "ordinary referents." Singly embedded occurrences, like those in the complement clauses in (6a) and (6b), express the "indirect senses" of expressions, which are modes of presentation that determine their ordinary senses as "indirect referents." The process is repeated in (8).

8.   Mary imagines that John believes that *the author of the Begriffsschrift was German*.

The occurrences in (8) of the words in

9.   John believes *that the author of the Begriffsschrift was German*

refer to the senses that occurrences of those words carry when (9) is not embedded – i.e. to the ordinary senses of 'John' and 'believes', plus the indirect senses of the words in the italicized clause. In order to do this, occurrences of 'John' and 'believe' in (8) must *express* their indirect senses (which are, of course, distinct from the ordinary senses they determine as indirect referents), while occurrences in (8) of the words in the italicized clause must *express* doubly indirect senses, which determine, but are distinct from, the singly indirect senses that are their doubly indirect referents. And so on, *ad infinitum*. Thus, Frege ends up attributing to each meaningful unit in the language an infinite hierarchy of distinct senses and referents.

But if that is so, how is the language learnable? Someone who understands 'the author was German' when it occurs in ordinary contexts doesn't require further instruction when encountering it for the first time in an attitude ascription. How, given the hierarchy, can that be? If s is the ordinary sense of an expression E, there will be infinitely many senses that determine s, and so are potential candidates for being the indirect sense of E. How, short of further instruction, could a language learner figure out which was *the* indirect sense of E? Different versions of this question have been raised by a number of philosophers from Bertrand Russell to Donald Davidson.