

Seminar 2

Challenges to Fregean Philosophy of Language and Mathematics

Existence and Generality

According to Frege, (1a,b) express the same proposition, which predicates the 2nd-order concept, *being true of at least one object* of the 1st-order concept *being so-and-so*.

- 1a. There is a (at least one) so-and-so.
- b. There exists a (at least one) so-and-so.

Since the proposition these sentences express makes an existence claim, Frege takes *existence* to be the 2nd-level concept expressed by ' $\exists x$ '. This is problematic.

Let C be the concept *being such and such*. C can then be truly predicated of something iff that thing is such and such. E.g., the concept *being bigger than an ant*, can be truly predicated of x iff x is bigger than an ant. Suppose existence is the concept *being true of at least one object*. What follows?

Frege does say that existence is a property of concepts.

"I have called existence a property of a concept. How I mean this to be taken is best made clear by an example. In the sentence 'there is at least one square root of 4' we have an assertion, not about (say) the definite number 2, nor about -2, but about a concept *square root of 4*; viz. that it is not empty." ("Concept and Object")

"By properties which are asserted of a concept I naturally do not mean the characteristics which make up the concept. These latter are properties of the things which fall under the concept, not of the concept. Thus "rectangular" is not a property of the concept "rectangular triangle"; but the proposition that there exists no rectangular equilateral rectilinear triangle does state a property of the concept "rectangular equilateral rectilinear triangle", it assigns to it the number naught. In this respect existence is analogous to number [in being a second-level concept]. Affirmation of existence is in fact nothing but denial of the number naught. *Because existence is a property of concepts the ontological argument for the existence of God breaks down.*" ("Foundations of Arithmetic")

Frege could have said: *There is no concept existence; when we use (1b) what we are really saying is that the concept being so-and-so is true of at least one object*. But would still cause (2) problems.

- 2a. 'Socrates' names something that no longer exists.
- b. Socrates is dead, and so no longer exists.
- c. The teacher of Plato is dead, and so no longer exists.

The particular quantifier ' $\exists x$ ' and the definite description operator 'the', aren't always restricted to ranging over existing things. Their range is context-sensitive. Sometimes they are restricted existing things, sometimes not. Sometimes they range over a small subset of existing things, as when a father uses (3) to truly describe the situation at one of his children's overnight parties.

- 3a. Everyone is asleep.
- b. The oldest boy is in charge of the others.

Consider three classes of concepts: (i) *being a human being, being a doctor, being employed by USC, being a dog owner, being located in Lima, existing* (ii) *being dead, being non-existent, and being something I will build in the future* (iii) *being loved/admired/hated/feared/forgotten, being referred to/talked about/surpassed/eclipsed, and being influential*. Which of these are true only of things that exist? Which are true only of things that don't exist? Which may be true of either?

Existence isn't a predicate because it is a concept and concepts aren't words. Perhaps the slogan really means that 'exists' doesn't function semantically as a predicate. But isn't a concept that assigns *truth* to

o iff o exists or (as Frege might prefer) to objects o such that for some y, $y = x$. Why shouldn't 'exist designate it'?

- 4a. Someone we have been talking about – namely Socrates/the teacher of Plato – is dead, and so doesn't exist, while someone else we have been talking about – namely Noam Chomsky (the famous MIT linguist) is still alive, and so does exist
- b. $\exists x \exists y (x \neq y \ \& \ \text{we have been talking about } x \ \& \ \text{we have been talking about } y \ \& \ x = \text{Socrates [the teacher of Plato]} \ \& \ y = \text{Noam Chomsky [the famous MIT linguist]} \ \& \ (x \text{ is dead} \ \& \ \sim x \text{ exist}) \ \& \ (y \text{ is alive} \ \& \ y \text{ exists}))$

Propositions

For Frege, propositions are meanings of complete sentences. Since these are what we believe/assume/know) etc., he called them thoughts. A sentence is true when the thought it expresses is true. So thoughts are the primary bearers of truth or falsity. *They are capable having truth values because they represent things as being certain ways.* When the things they represent are the way they are represented to be the thoughts are true. When the things they represent aren't as they are represented to be, the thoughts are false. Since thoughts are meanings, Frege took them to be structured complexes the constituents of which are meanings of words and phrases of the sentences that express them. Russell held similar views, which led to the problem of *the unity of the proposition*.

Frege ("On Concept and Object")

"In logical discussions one quite often needs to assert something about a concept, and to express this in the form usual for such assertions – viz., to make what is asserted of the concept into the content of the grammatical predicate. Consequently, one would expect that the reference of the grammatical subject would be the concept; *but the concept as such cannot play this part, in view of its predicative nature; it must first be...represented by an object.* We designate this object by prefixing the words 'the concept', e.g. 'The concept *man* is not empty.' Here the first three words are to be regarded as a proper name, *which can no more be used predicatively than 'Berlin' or 'Vesuvius.'* When we say 'Jesus falls under the concept *man*,' then, *setting aside the copula, the predicate is: 'someone falling under the concept man' and this means the same as 'a man.'* But the phrase 'the concept *man*' is only part of this predicate."

"We may say in brief, taking 'subject' and 'predicate' in the linguistic sense: A concept is the reference of a predicate; An object is something that can never be the whole reference of a predicate, but can be the reference of a subject."

"Somebody may think that this is an artificially created difficulty; that there is no need at all to take account of such an unmanageable thing as what I call a concept; that one might...regard an object's falling under a concept as a relation, in which the same thing could occur now as object [i.e. as referent of a singular term], now as concept [i.e. as referent of a predicate]...This may be done; but anybody who thinks the difficulty is avoided in this way is very much mistaken; it is only shifted. *For not all the parts of a thought can be complete; at least one must be 'unsaturated', or predicative; otherwise they would not hold together.* For example, the sense of 'the number 2' does not hold together with that of the expression 'the concept *prime number*' without a link. We apply such a link in the sentence 'the number 2 falls under the concept *prime number*'; it is contained in the words 'falls under,' which need to be completed in two ways – by a subject and an accusative; and *only because their sense is thus 'unsaturated' are they capable of serving as this link.* I say that such words or phrases stand for a relation. We now get the same difficulty for the relation that we were trying to avoid for the concept. For the words 'the relation of an object to the concept it falls under' designates not a relation but an object; and *the three proper names 'the number 2', 'the concept prime number', 'the relation of an object to a concept it falls under', hold aloof from one another just as much as the first two do by themselves; however we put them together we get no sentence.* It is thus easy for us to see that the difficulty arising from the 'unsaturatedness' of one part of the thought can indeed be shifted, but not avoided."

Russell (*Principles of Mathematics*)

"Consider, for example, the proposition "A differs from B." The constituents of this proposition, if we analyze it, appear to be only A, difference, B. *Yet these constituents, thus placed side by side, do not reconstitute the*

proposition. The difference which occurs in the proposition actually relates A and B, whereas the difference after analysis is a notion which has no connection with A and B. It may be said that we ought, in the analysis, to mention the relations which difference has to A and B, relations which are expressed by is and from when we say A is different from B. These relations consist in the fact that A is referent and B relatum with respect to difference. But A, referent, difference, relatum, B, is still merely a list of terms, not a proposition. A proposition, in fact, is essentially a unity, and when analysis has destroyed the unity, no enumeration of constituents will restore the proposition. The verb, when used as a verb, embodies the unity of the proposition, and is thus distinguishable from the verb considered as a term, though I do not know how to give a clear account of the precise nature of the distinction.”

We begin with a question. *How does a sentence differ from a mere list of its words?* We answer that a sentence S has a complex syntactic structure. The position of the words in the structure tells us what a user of S predicates of what. Because of this, S represents the things designated by the arguments of the predicate as having the property it expresses. What these things are and what property is predicated of them is determined by the meaning of S. Thus, the explanation of *the unity of a sentence* (as combination of words with representational content) depends on the proposition itself being a unified combination of expression meanings that has representational content. For example, the proposition *that difference is different from identity* is a unified combination of the meanings of ‘difference’, ‘is different from’, and ‘identity’. The proposition represents the relations *difference* and *identity* as being different. The problem of the unity of the proposition is to explain how this content read off the proposition.

For Frege, some senses (meanings) are by nature predicative – whenever they occur in a proposition the concept (function from entities to truth values) they determine is either (a) predicated of the objects determined by non-predicative senses (expressed by singular terms), or (b) it is the argument of a higher-level concept determined by a higher-level predicative sense. For example, in the proposition *that Lima is different from Los Angeles*, the sense of the predicate ‘is different from’ is inherently predicative, while the senses of ‘Lima’ and ‘Los Angeles’ are inherently non-predicative. Because of this, Frege thinks, we take the proposition to predicate *being different from* of the pair of cities, and we recognize the proposition as representing them as different.

Problems: (i) How can we accept the conclusion this leads Frege to – namely that the concept *horse* is not a concept? (ii) How does this account of the proposition *that difference is different from identity* – in which all the constituents ought, for Frege, to be inherently predicative? (iii) How is Frege’s account of propositional unity consistent with his account of quantification, according to which a concept is *the referent of a lower-order predicate* and *the subject of the assertion made by using the quantified sentence*? Since these italicized phrases are singular terms, to use them, as Frege must, to designate concepts of which some 1st-order predication is made is to treat them as objects, in violation of his own view.

Agents, Propositions, and Intentionality: The Order of Explanation

Propositions – i.e., things believed, asserted, and known -- represent things as being certain ways, and so have truth conditions. When agents entertain or accept propositions, they mentally represent things as being certain ways. These two facts are related. So one can be used to explain the other. Frege tries to use the former fact to explain the latter: For him, an agent A mentally represents something as being a certain way *because* A cognizes a proposition that does so. This is what led to his problems.

The alternative view starts with the idea of *an agent* representing an object as being a certain way. E.g., we take it as basic that A mentally represents o as being red when A visualizes o as red, imagines o as red, or in any other way cognizes or perceives o as red. Next look for a kind of entity P and a relation R that guarantee *agents* who bear R to something of kind P thereby represent things as being some way. With such P and R, we can *explain* the intentionality of things of kind P by *deriving* it from the intentionality of

agents who bear R to them. If for A to bear R to an item p^* of kind P *just is* for A to represent o as *being red*, then p^* may be deemed true iff o is red.

So, what are propositions and what is it to entertain one? Perhaps *propositions are repeatable, purely representational, cognitive acts or operations; to entertain one is not to cognize it but to perform it*. When I perceive or think of o as red, I *predicate* the property *being red* of o, which is to *represent o as red*. This act *represents* o as red in a sense similar to the derivative senses in which acts can be *insulting* or *irresponsible*. An act is insulting when to perform it is for one to insult someone; it is irresponsible when to perform it is to neglect one's responsibilities. A similar derivative sense of *representing* allows applies to an agent's sayings or cognitions. When to perceive or think of o as P is to represent o as it really is, we identify the cognitive act of representing o as P plus a property it has when the cognition is accurate. The entity is a proposition; the property is truth, which the act has iff to perform it is to represent o as o really is.

This approach was not open to Frege because it derives the intentionality of propositions from the prior intentionality of minds, whereas he took the opposite approach.

Truth

Just as Frege wrongly believed that 'exist' isn't a predicate of objects so he wrongly believed that 'is true' isn't a predicate at all. Here is a passage from "On Sense and Reference."

One might be tempted to regard the relation of the thought to the True not as that of sense to reference, but rather as that of subject to predicate. *One can indeed say: 'The thought, that 5 is a prime number, is true.'* But closer examination shows that nothing more has been said than in the simple sentence '5 is a prime number.' *The truth claim arises in each case from the form of the declarative sentence, and when the latter lacks its usual force, e.g., in the mouth of an actor upon the stage, even the sentence 'The thought that 5 is a prime number is true' contains only a thought, and indeed the same thought as the simple '5 is a prime number.'* It follows that the relation of the thought to the True may not be compared with that of subject to predicate. Subject and predicate (understood in the logical sense) are indeed elements of thought; they stand on the same level for knowledge. By combining subject and predicate, one reaches only a thought, never passes from sense to reference, never from a thought to its truth value...A truth value cannot be part of a thought, any more than, say, the Sun can, for it is not a sense but an object.

Even if one accepts his view that the True is an object to which all true sentences refer, one can still define a truth predicate: for all thoughts T, *T is true* iff T determines the True (where *determination* holds between a sense s and entity x when expressing s determines that an expression refers to x). With this definition, the thought (5a) consists of the indirect sense of (5b) (which picks out the thought (5b) customarily expresses) plus the sense of 'is true', which designates the concept *determines the True*.

- 5a. The thought that 5 is a prime number is true.
- b. 5 is a prime number.

By contrast, the thought expressed by (5b) predicates the concept *prime number* of 5. So the thoughts (5a) and (5b) are different, even though we know a priori that they are necessarily equivalent. *Anyone who understands the two sentences knows that the thought expressed by (5b) is the thought of which truth is predicated by the thought expressed by (5a). One immediately sees that in assertively uttering either sentence one asserts, and thereby commits oneself to, both thoughts.* So no new information is added by an assertion of (5a) that is not already made available by an assertion of (5b). Nevertheless the content of the truth predicate isn't empty, as shown by cases like 'You can believe whatever he says, because what he asserts is always true'.

We can now definitively establish that 'is true' functions as a genuine predicate of propositions. The key insight is found in Frege's 1918 paper 'The Thought'.

“we cannot recognize a property of a thing without at the same time finding the thought *this thing has this property* to be true. So, with every property of a thing there is tied up a property of a thought, namely truth. It is also worth noticing that the sentence ‘I smell the scent of violets’ has just the same content as the sentence ‘It is true that I smell the scent of violets.’ So it seems, then, that nothing is added to the thought by my ascribing to it the property truth. And yet is it not a great result when the scientist after much hesitation and laborious research can finally say ‘My conjecture is true’? The meaning of the word ‘true’ seems to be altogether *sui generis*.”

Here, Frege contrasts two linguistic environments in which the truth predicate occurs.

Environment 1: *It is true that S / The thought that S is true*

Environment 2: *My conjecture / everything John said / something Mary believes is true*

When we use the truth predicate in Environment 1 – e.g., in [it is true that S] – we don’t add anything significant to what we express with S alone. But this doesn’t show that the thoughts are identical, because it is consistent with the view that the thoughts are different but trivially, equivalent—which explains why explicitly committing oneself to one by accepting a sentence that expresses it implicitly commits one to the other as well). By contrast, when the truth predicate is used in Environment 2, it does add something to the thought expressed by the sentence containing it – and so is not dispensable in the way it is in Environment 1.

In Environment 2, we predicate truth of thoughts, we don’t explicitly assert or display. E.g., we might assert that everything John reported is true, because we know his character and intellect, even if we don’t know what he reported, or believe the conjunction of all he, in fact, asserted. Similarly, we can claim that *every thought or its negation is true*, without having to produce an infinite list of disjunctions. *In such cases, we use the truth predicate to say something we would not be able to say without it.* If we never wished to say of a thought that it is true without displaying it, as we do when we say [It is true that S] or [The thought that S is true], then we wouldn’t need a truth predicate. *In short, Environment 2 is the reason we have one.*

Environment 1 is useful in explaining what truth is. In explaining what it is for the thought that violets are flowers to be true, it is useful to point out that it is true iff violets are flowers. In general, T and the thought *that T is true* are necessary and a priori consequences of each other, and any warrant for asserting, denying, believing, doubting, toward T, or toward the thought *that T is true*, is warrant for taking that attitude toward the other.

Sense, Reference, Identity

It is curious that Frege’s distinction between sense and reference, which applies to expressions of many grammatical categories, and to sentences of all grammatical forms, is presented in “On Sense and Reference” as if it had something special to do with identity sentences. In the first paragraph he presents what seems to be an argument that *identity* isn’t a relation between and *object* and itself.

- P1. If identity were a relation between objects, then in those cases in which the statement is true, knowing what the identity relation is plus knowing what objects *n* and *m* are is sufficient for knowing *that n = m*.
- P2. Anyone who knows what identity is knows that it holds between each object and itself, and when *n* and *m* are the same object, anyone who knows what object *n* is and also what object *m* knows that they are the same object.
- C1. So, if identity is a relation between objects, then *knowing that n = m* is trivial.
- P3. In many cases, such knowledge *isn’t* trivial.
- C2. So identity isn’t a relation between objects.

Although the argument can appear persuasive, it must be unsound, because if it were sound, then virtually no familiar properties or relations would be properties or relations of objects.

Suppose we replace *identity* in the argument with any other relation known a priori to be reflexive -- i.e. to hold of each object and itself (whether or not it holds of other objects as well). Making this change -- e.g. by substituting *not larger than* for *identity* -- allows us to conclude the *not larger than* isn't a relation between objects. But that can be so only if *larger than* isn't a relation between objects. The point generalizes. When $F(x)$ and $G(y)$ are formulas (with only 'x' and 'y' free) designating concepts that can be known a priori not to be satisfied by the same object, the concept designated by $[\sim [F(x) \ \& \ G(y)]]$ will be a relation that is knowable a priori to be reflexive. So, if the identity argument were sound, this concept *wouldn't be* a relation on objects, in which case neither the relation designated by $[F(x) \ \& \ G(y)]$ nor *the concepts designated by $F(x)$, and $G(y)$, could be properties of objects*. Since this threatens the idea that *any* concepts hold of objects, it is a *reductio ad absurdum* of the argument that *identity* isn't a relation on objects.

This means that at least one of the premises in the identity argument is false. When 'n' and 'm' in the argument are replaced with singular definite descriptions, P1 and P2 are false. When 'n' and 'm' are ordinary proper names, there is a way of construing the identity propositions expressed containing them in which only P3 is false and there is another way of construing those propositions in which P3 is true but P1 and P2 are false. See Soames *Rethinking Language, Mind, and Meaning*.

It remains to be explained how, on Frege's own view, identity should be seen as a relation between objects, and how properties like *being German* should be seen as properties of objects. The explanation of the former can be generalized from the explanation of the latter. Let the descriptions in (6) pick out the same thing.

- 6a. The F is G.
- b. The H is G.

On the Fregean analysis of definite descriptions, 'the' designates the function f_{the} which takes a concept C as argument and assigns o as value iff o is the only thing of which C is true. Entertaining the thought expressed by (6a) involves (i) applying f_{the} to the concept *being F* and (ii) predicating the concept *being G* of the result. Entertaining the thought expressed by (6b) involves (iii) applying f_{the} to the concept *being H* and (iv) predicating the concept *being G* of the result. These thoughts are different because (i) and (iii) are different, even though, in our example, they result in the same object. (i) involves thinking of the concept *being F*, and hence having it in mind under a certain mode of presentation, while taking it as argument of f_{the} , whereas (iii) involves thinking of the concept *being H*, and hence having it in mind under a different mode of presentation, while taking it as argument of f_{the} . So, although the same thing (*being G*) gets predicated of the same object, the manner in which this is done is different in the two cases, reflecting the different constituents of the two Fregean thoughts, and the designations they determine. Applying this idea to identities allows one to explain how the thoughts expressed by [the F = the F] and [the F = the G] can differ, even if they predicate the identity relation of the same objects.

Non-Transparent Thoughts

In addition to taking propositions to be both meanings of sentences and the things we believe and know, Frege took them to be transparent to us. When two expressions (or two sentences) mean the same thing, Frege thought that anyone who understands them will know they mean the same thing. He also thought that when one of two sentences, or one of two thoughts, was the negation of the other, anyone who understood the sentences, or entertained the propositions, would see they are contradictory. This led him to conceive of senses of expressions (constituents of propositions) as satisfying a transparency condition: anyone acquainted with several occurrences of the same sense (propositional constituent) will always recognize them as the same. This leads to problems

Consider (7), which is true despite the fact that the ways the ancients thought of Venus (i.e. the senses they used to pick it out) were different morning and evening.

7. *There is a planet (Venus) of which the ancients said, and believed, when they saw it in the morning, that it was visible only in the morning, and of which they said, and believed, when they saw it in the evening, that it was visible only in the evening*

How can (7) be true. Here is a possible explanation. The italicized phrase is a quantifier binding occurrences of 'it' (which functions as a variable). On the standard analysis, 'There is an x such that ... x...' is true iff there is some object o such that '...x...' is true when o is assigned as referent of 'x'. Next we take what a variable contributes to the thought expressed by a formula containing it to be its referent o (relative to an assignment). We further take one to believe that thought iff one believes, of o, that it has the properties given by the thought, *where this belief doesn't require thinking of o in any one specific way*. This will explain the truth of (7).

(7) is true iff '*the ancients said, and believed, when they saw x in the morning, that x was visible only in the morning, and of which they said, and believed, when they saw x in the evening, that x was visible only in the evening*' is true', relative to an assignment A of Venus to 'x'. That, in turn, is true iff the ancients (i) asserted and believed the thought p expressed by '*x is visible only in the morning*' relative to A, when they saw Venus in the morning, and (ii) asserted and believed the thought q expressed by '*x is visible only in the evening*' relative to A, when they saw Venus in the evening. *Here, p is the non-Fregean thought containing Venus that attributes to it the property of being visible only in the morning, while q is the corresponding thought that attributes to it the property of being visible only in the evening. Believing these thoughts (called singular propositions) doesn't require thinking of Venus in one particular way. There are, of course, some constraints on how one must think of o in order to believe a singular proposition about it. It is not enough to think "the F, whatever it may be..." for absolutely any F that happens to pick out o. But these constraints leave room for believing one thing about Venus by virtue of thinking of it in one way, and believing a different, inconsistent, thing about it by virtue of thinking of it in another way -- without being able to notice the inconsistency because it is not transparent that the two ways of thinking about Venus are ways of thinking of the same thing. This is what (7) correctly reports. In cases like this, we report agents' attitudes toward objects in a way that abstracts away from the precise manner in which they think about them. This is useful because we often don't know what those precise manners are.*

Now consider an extension of the case in which we imagine Venus as a sentient being observing the ancients on Earth and assertively uttering sentence (8) to her companion, Mercury.

8. The ancients asserted and believed, when they saw *me* in the morning, that *I* was visible only in the morning, and they asserted and believed, when they saw *me* in the evening, that *I* was visible only in the evening.

Here, Venus says something true and nothing false, even though neither the way she thinks of herself, nor the way Mercury thinks of her, could have been modes of presentation (senses) of Venus for the ancients. This is not a problem if some propositions are not transparent. Examples similar to (8) can be constructed using other indexicals – including 'you', 'he'/'she', 'this', and 'that'. A case for non-transparent propositions can also be made on the basis of examples like (9).

9. *Bill* fooled *Mary* into thinking that *he* wasn't *Bill*.

Russell's Paradox

At the turn of the 20th century, Bertrand Russell was, like Frege, trying to reduce arithmetic to logic. He also conceived of numbers as sets, though in a way that differed slightly from Frege's. For Frege the number 2 is the set of *concepts the extensions of which* contain some distinct x and y, and only them; for

Russell concepts aren't mentioned, and the number 2 is the set of *sets* that contain some distinct x and y , and only them. Both also recognized a universal set of all objects, where sets themselves count as objects. Russell noticed a problem that arose in his study of Cantor's proof that for any set s , the set of all subsets of s (the *power set* of s) is larger than s (in the sense that whereas s can be put in 1-1 correspondence with a subset of the power set of s , the power set of s can't be put in 1-1 correspondence with any subset of s). Russell adapted Cantor's proof to show that there is no universal set. He also found a problem for himself and Frege.

Russell's logic, like Frege's can be taken to have an unrestricted axiom schema of comprehension for sets.

Comp: $\exists y \forall x (Fx \leftrightarrow x \in y)$

Instances of this schema arise from replacing Fx with any formula containing only ' x ' free. The idea is that whatever formula replaces Fx , there must be a set of all and only those things of which the formula is true. In some cases, this set will be empty. But that isn't a problem, since the empty set is (we may imagine) still a set. To think of Comp as a principle of logic is to think that talk about x 's being so-and-so is interchangeable with talk about it being in the set of things that are so-and-so.

Suppose we replace Fx with the formula ' $\sim x \in x$ '. Doing this gives us (10) as a logical axiom.

10. $\exists y \forall x (\sim x \in x \leftrightarrow x \in y)$

There is a set of all and only those things that are not members of themselves.

Let us introduce a new symbol ' R ' that names this set. (11) must be true, if (10) is.

11. $\forall x (\sim x \in x \leftrightarrow x \in R)$

Everything is such that it is a member of R iff it is not a member of itself

Since everything includes absolutely all things, (11) includes R . So, (12) must be true, if (11) is.

12. $(\sim R \in R \leftrightarrow R \in R)$

R is a member of R iff R is not a member of R .

Since (12) is a contradiction, it can't be true. Hence neither (10) nor Comp are laws of logic.

After discovering this problem, Russell examined Frege's work hoping it might contain a solution. After discovering that it didn't, and that the contradiction in his logic also arose in Frege's, he informed Frege in a letter.

Here is how the problem arose for Frege. Numbers, for him, are extensions of concepts, extensions are sets, and sets are objects. Consider the first-order concept *being the extension of a concept that is not true of its own extension*. Although this concept – call it C – can't be applied to itself, it can be applied to its extension. If applying it yields truth, then C is true of its own extension; if applying it yields falsity, then C is not true of its own extension. Since all concepts are defined for all arguments, one of these results must hold. *Suppose it is the latter, i.e., that applying C to its own extension yields falsity*. Then C is *not* true of its own extension. Then, by definition of C , C 's extension is the extension of a concept (namely C) of which that concept isn't true. This guarantees that *C is true of its own extension*, which contradicts our earlier supposition, that C isn't true of its own extension. *So we are forced to conclude that, in fact, C does assign truth to its own extension, and so C is true of its own extension.*

But then there must be *some concept*, call it C^* , the extension of which = the extension of C , where C^* isn't true of that extension but C is. Frege's Axiom V tells us this is impossible.

Axiom V: For all (first-level) concepts P and Q , the extension of P (the class of things falling under P) = the extension of Q (the class of things falling under Q) iff $\forall x (Px \leftrightarrow Qx)$.

For if (i) the extension of $C =$ the extension of C^* and (ii) P_C designates C and P_{C^*} designates C^* , then, by Axiom V, $[\exists x (P_C x \ \& \ \sim P_{C^*} x)]$ can't be true, which means that there is no object x such that C is true of x while C^* isn't. Since the extension of C is an object, it can't be the case that C is true of it while C^* isn't. Thus we have a contradiction; Frege's logical system is inconsistent.

Frege, saw the point and tried to amend his system by adding an appendix to *The Basic Laws of Arithmetic*, which was already in press. He came up with a hasty repair, in which he didn't have much confidence. His repair addressed two questions *Do all concepts have sets as extensions* and *Is it true that if two concepts have the same extensions, either both are true, or both are false, of that object?*

The repair of Axiom V answered the first question, 'yes', which meant that he retained the right-to-left half of Axiom V. He answered the second question 'no', which meant that he had to modify the left-to-right half. Here is the idea. One takes the extension of a (1st-order) concept C to include all objects of which C is true, *except the extension of C itself (so the extension of a concept is never a member of itself)*. This guarantees that when C and C^* are true of the same objects, their extensions will be identical (which is the right-to-left half of Axiom V). But it is possible for the extensions of C and C^* to be identical, even though C is true of their common extension, but C^* isn't. This is what is ruled out by the modification of the left-to-right half of V. The new axiom is:

Axiom V'

For all (1st-order) concepts P and Q , the extension of $P =$ the extension of Q iff for all x such that $x \neq$ the extension of P and $x \neq$ the extension of Q , $(Px \leftrightarrow Qx)$.

Consequences of V' include (i) (the original right-to-left half of V), (ii) (the right-to-left half of V'), and (iii) (the left-to-right half of V')

- (i) For all (1st-order) concepts P and Q , if $\forall x (Px \leftrightarrow Qx)$, then the extension of $P =$ the extension of Q .
- (ii) For all (1st-order) concepts P and Q , if for all x such that $x \neq$ the extension of P and $x \neq$ the extension of Q , $(Px \leftrightarrow Qx)$, then the extension of $P =$ the extension of Q .
- (iii) For all (1st-order) concepts P and Q , if the extension of $P =$ the extension of Q , then for all $x \neq$ the extension of P and $x \neq$ the extension of Q , then $(Px \leftrightarrow Qx)$.

But removing one argumentative route to inconsistency is not enough. In addition to being far from self-evident in the sense that Frege requires to serve his ambitious epistemological goals, V' leads to other problems. As Michael Dummett noted, it blocks his former proof that every natural number has a successor, which he surely would have noticed had he had time prior to publication to check the proofs in which Axiom V plays a crucial role. Worse, it doesn't save Frege's system from falsification by Russell's paradox. The key point, shown by Quine in 1955, is that V' has the consequence that there is at most one object. See pp. 126-28 of Volume 1 of *ATP*. Although not itself a contradiction, this result is absurd, and would block any attempt to ground mathematics along Fregean lines. Nor is contradiction really avoided, since Frege's assumption that the True and the False are distinct objects is inconsistent with the consequence Quine showed to be derivable from Axiom V'. Thus, Frege's rushed attempt to modify his original inconsistent system to avoid paradox fails.

Although no such demonstration appeared during Frege's lifetime, no one adopted his modified system, and even he soon found it to be inadequate. As a result, the founder of modern logic largely dropped out of the next great stage of its development, abandoning his work on formal logic forever, and doing very little philosophical logic between 1903 and 1918. He showed little or no interest in the development of set theory, and repudiated the idea that arithmetic could be derived from logic at all. In short, the failure of his logicist project, brought on by Russell's paradox, was a blow from which he never recovered. In addition to presenting a mathematical difficulty for which he didn't have a solution, *the paradox struck at the heart of his ambitious epistemology of logic as the indispensable, self-evidently obvious bedrock*

on which all knowledge of number is founded. This was the grand vision that could not be saved. It is not just that the repair was ad hoc and demonstratively defective. More profoundly, the centrality and severity of the problem cast doubt on the idea that any repair, even if technically adequate, would have the self-evidence required to preserve his epistemological vision.

Focusing on the failure of Frege's hoped for reduction of mathematics to logic, and on downward trajectory of the last half of his career after such an impressive start, we should resist the temptation to dwell more on the shadow cast by his failure than by the illumination provided by his early success. *It is in the nature of philosophy for our imaginative reach to exceed our demonstrative grasp.* Our greatest thinkers are guided by powerful and compelling visions that include elements that go beyond what they can demonstrate, or, in some cases, even rationally defend. Although this leads to dashed hopes and failed expectations, it is likely that, if our greatest philosophers had been less ambitious, they would not have left us with the real philosophical progress we now enjoy. Frege's vision of the realms of logic, language, and mathematics is among the most compelling and original we have ever known – and also among the most fruitful. One of the greatest philosophers of mathematics of all time, he was, along with Gödel and Tarski, among those most responsible for the stunning development of modern symbolic logic – itself one of the paramount intellectual achievements of the past century and a quarter. Even though Frege did not contribute to the development of set theory after the turn of the century, his reductionist project remained an inspiration for the use of set theory as a foundation for mathematics, including the now standard reductions of arithmetic to set theory. When it comes to the study of language, and the information it carries, we still have a long way to go. But it was Frege whose analysis of quantification and whose compositional theories of sense and reference put us on the path. Even today, it is still Frege – and Russell, Tarski, Carnap, Kripke, Montague, and Kaplan – to whom we look for key elements of our scientific framework for studying language. Centuries from now, when our descendants reap the benefits of the knowledge flowing from an advanced science of language and information, they will remember Frege as one of the giants who made it possible.